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CONTENTS

	PAGE
Afzal Ahmed: Linear equations in a quaternionic field ...	65
Chandrasekharan, K.: Partially ordered sets and symbolic logic	14— 24
Chowla, S.: Contributions to the theory of the construction of balanced incomplete block designs ...	82— 85
Chowla, S. and Daljit Singh: A perfect difference set of order 18	85
Daljit Singh and Chowla ,, ,,	,,
Gangadharan, K. S.: On some identities and a generalization of Wilson's theorem	75— 77
Hamid, S. A. On the equatorial coordinates of a star and the rectangular coordinates of its image on the photographic plate	80— 81
Kesava Menon, P.: An extension of a theorem of Steiner	78— 79
Krishnan, V. S.: Partially ordered sets and projective Geometry	7— 14
Kosambi, D. D.: Garret DavidBirkhoff	116—120
Lakshmanamurti, M.: Analytical proof of Gaskin's theorem	106
Madhava Rao, B. S.: Modern Algebra and the theory of elementary particles	30— 58
Moorty, T. N.: Lightning elimination	107—108
Narasinga Rao, A.: On the metric Geometry of a cyclic N-point — I	91— 97
Narasinga Rao, A. and Venkataraman, M.: On the Clifford and Grace chains	98—100
Noronha, F. J.: Equations reducible to Clairaut's form ...	70— 72
———— On reciprocal screws	73— 74
Pankajam, S. (Miss): Group operations in certain distributive lattices	25— 29
Parameswaran, S.: Some properties of the Eulerian and Bernoullian numbers	104—105
Puri, A. S.: Transcendence of decimals	88— 90
Racine, C.: The method of Leray-Schauder	63— 64
Rangachariar, V. and B. Singh,: On the radical conic of two central conics	86— 87

PAGE

Satyanarayana, K. : Sequences whose kth differences form a geometrical progression 109
Shah, S. M. : A note on Meromorphic functions ... 67— 70
Singh, B. and Rangachariar, V. : On the radical conic of two central conics ... ... 86— 87
Srinivasan, A. K. : Remarks on two notes in Vol. XI ... 103
Subba Rao, M. V.; Fractional powers of linear operators 64
Subrahmanian, H. G. : Integer roots by inspection ... 107
Suryanarayanamurthy, A. : Topology of ordered Spaces ... 66
Tiruvenkatachari, V. R. : Connection between spin and statistics ... ... 59— 62
Venkataraman, M, and Narassinga Rao, A. : On the Clifford and Grace chains ... ... 98—100
Vaidyanathaswami, R. : Partially ordered sets ... 1— 6
Yaji. : On the formula $\sum PA^2 = \sum GA^2 + 3PG^2$ ... 108—109
Announcements and news 121
Accounts of I. M. S. for 1942 and 1943 ... ... 122—123
Reviews :
Gorakh Prasad : *Text-book on differential calculus*; ,, *Text book on integral calculus*; ,, *Key to differential calculus*; ,, *Calculus for Intermediate students* ... 66
B. Segre, : *Non-singular cubic surfaces* ... 111
J. E. Littlewood, : *Lectures on the theory of functions* 112
E. H. Neville : *Jacobian Elliptic functions* ... 112—113
S. K. Abhayankar : *Algebra for Intermediate*; S. Narayanan : *Dynamics for B.A., B.Sc., and F. E. classes* ... 113—114
Barlow and Bryan : *Elementary Mathematical astronomy* 115

ERRATA FOR VOLUME XII

Page	*line*	*For*	*Read*
78	7	concurrent B′	concurrent
90	22	once	ones
98	15	21)	(21)
98	last	1,2,3,4	(1)′,(2)′,(3)′ and (4)′

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SYMPOSIUM ON PARTIALLY ORDERED SETS

CHAIRMAN: DR. R. VAIDYANATHASWAMY

Opening Speech on "Partially Ordered Sets" By the Chairman

I. The system of real numbers is *linearly* ordered, or *simply* ordered by the asymmetric transitive relation $<$ (less than); the relation $\leqslant$ satisfies

P_1 $a \leqslant a$ (reflexivity)

P_2 $a \leqslant b$ and $b \leqslant c$ imply $a \leqslant c$ (transitivity)

P_3 $a \leqslant b$ and $b \leqslant a$ imply $a = b$

P_4, Either $a \leqslant b$ or $b \leqslant a$

An abstract set K is said to be partially ordered by a binary relation $\leqslant$, if the relation has the properties P_1 P_2 P_3. If the relation has also the property P_4, then the partial order reduces to *simple*, or *linear* order. It thus appears that partial order is a generalisation of linear order. A simple example of a partially ordered set is the set of points of Euclidean space of two (or more) dimensions, when the ordering relation is defined by

$$(a, b) \leqslant (c, d) \text{ if } a \leqslant c; \ b \leqslant d.$$

If the elements of the set K are subsets of a set S, then K is always partially ordered by the relation of set-inclusion $\subset$ between its elements. For the relation $\subset$ of set-inclusion between subsets of a set S is easily seen to possess the properties P_1, P_2, P_3; in general it does not possess the property P_4.

The elements 0, 1 of a partially ordered set K, defined by $0 \subset$ every element of $K \subset 1$, are called *units* of K. Of course K may or may not have units.

II. The set K_S of all subsets of a set S, partially ordered by the relation of set-inclusion between its elements, has the property that any subset of K_S has a set-union and a set-intersection. This is

generalised for a general partially ordered set K, by the notions of *lattice-sum* and *lattice-product*. The lattice-sum $\sum(P)$ and the lattice-product $\pi(P)$ of any subset P of a partially ordered set K are elements of K defined by :—

(1) every element of $P \subset \sum(P)$;
every element of $P \subset x$ implies $\sum P \subset x$.

(2) $\pi(P) \subset$ every element of P;
$x \subset$ every element of P implies $x \subset \pi(P)$.

The elements $\sum(P)$, $\pi(P)$ may or may not exist, but if they exist they are seen by P_3 to be unique. The lattice-sum and lattice-product are dual conceptions and change over into each other when the partial order is reversed. They possess a general associative and commutative property. The partially ordered set K is said to be a *lattice* if lattice-sums and lattice-products of any finite number of elements exist, and a complete lattice if all lattice-sums and products exist.

III. *Distributivity.*

Set-union and set-intersection are related by mutual distributivity in set-algebra; *i.e.*

$$A \cdot (B+C) = A \cdot B + A \cdot C$$
$$A + (B \cdot C) = (A+B) \cdot (A+C).$$

where A, B, C are sets, + and $\cdot$ denote set-union and intersection.

The corresponding property fails for the lattice. We have on the other hand,

if $a \cdot (b+c) = a \cdot b + a \cdot c$ always, then $a + (b \cdot c) = (a+b) \cdot (a+c)$, where a, b, c are elements of the lattice K, + and $\cdot$ denoting the lattice-sum, and lattice-product; and *vice versa*.

Proof. Since $a \cdot (b+c) = a \cdot b + a \cdot c$, for all a, b, c, we have

$$\begin{aligned}(a+b)\cdot(a+c) &= (a+b)\cdot a + (a+b)\cdot c\\ &= a + ac + bc, \text{ by definition of lattice product}\\ &= a + (b \cdot c).\end{aligned}$$

Similarly if we are given that $(a+b)\cdot(a+c) = a + b \cdot c$ for all a,b,c, we have:

$$\begin{aligned}a\cdot b + a \cdot c &= (a \cdot b + a)\cdot(a \cdot b + c)\\ &= a\cdot(a+c)\cdot(b+c) = a\cdot(b+c).\end{aligned}$$

These results may be expressed by saying that *the distributive property* is *self-dual.* Lattices possessing this distributive property are called *distributive lattices.* A restricted form of the distributive property is also self-dual; namely, if we are given that

$$a{\cdot}(b+c)=b+a{\cdot}c \quad \text{whenever } b\subset a,$$

then it follows immediately that

$$b+a{\cdot}c=a{\cdot}(b+c), \quad \text{whenever } b\subset a.$$

Lattices possessing this restricted distributive property are called *modular lattices.*

In contrast to the distributive law for finite sums and products, the distributive law for infinite sums does not imply the corresponding law for infinite products.

The simplest example of the distributive lattice is the chain lattice. If the partial order in K satisfies P_4, it reduces to linear order, and K is called a chain. A chain K is automatically a lattice and a distributive lattice (the lattice sum and lattice product of two elements a, b in K being the greater and smaller of a, b).

The other important example of a distributive lattice is the Boolean Algebra, which may be roughly described as a lattice having a structure which is formally similar to the lattice of all subsets of a set.

IV. *Product-Complements in a Distributive lattice.*

The notion of *set-complement* in the algebra of all subsets of a set may be generalised into the notion of 'Product-Complement' in a distributive lattice with units, 0,1. a' is the product-complement of a, if $aa'=0$ and if $ax=0$ implies $x\subset a'$. The product-complement of a is unique if it exists. Also if $a\supset b$, $a'\subset b'$. If every element of the distributive lattice has a product complement, then the following properties hold

$$(a+b)'=a'{\cdot}b'; \quad a'''=a'; \quad (a{\cdot}b)'=(a'+b')''.$$

An element a of the distributive lattice is said to be *normal* if $a=a''$; it is said to be *simple* if $a+a'=1$. The simple elements are necessarily normal; the normal elements are necessarily of the form b'. A Boolean Algebra may be defined as a distributive lattice with product-complements, in which every element is simple. The normal elements of the distributive lattice closed for product-complementation, form a Boolean Algebra, as may be proved from the above properties of product complements.

V. *Ideal-theory of the distributive lattice.*

If the distributive lattice D with 0 is not closed for product-complements, it can be extended into a lattice with the required property by the introduction of *Ideals.* According to M. H. Stone, an ideal of D is any sub-set P of D, with the properties:

(1) If a is in P and $b \subset a$, then b is in P.

(2) If a and b are in P, $a+b$ is in P.

In particular the set of elements $\subset$ an element a is an ideal, which is called the *principal ideal* defined by a. The set D′ of ideals of D when partially ordered by the relation of set-inclusion, is a complete lattice with completely distributive sums, and is closed for product complements. Moreover the lattice sum and product in D′ of principal ideals defined by the elements $a_1\ a_2 \ldots a_n$ of D are the principal ideals defined by the sum and product respectively of a_1, a_2, a_n. Also the product-complement in D′ of the principal ideal defined by a is the set of elements x such that $ax=0$; in particular, if a has the product-complement a', it is the principal ideal defined by a'.

A basic theorem in ideal-theory is Stone's factorisation theorem that *any ideal is the product of all prime ideals* containing it. A prime ideal is defined as an ideal which does not contain a product of two elements without containing either of them.

The ideals which we have so far spoken of are 'μ-ideals'. In a distributive lattice D which contains 1, the α-ideals are defined dually as the subsets P which have the property that whenever they contain two elements a, b, they contain the product ab, as well as any element $\supset a$. If D contains 0, 1, and P is any prime μ-ideal the set of elements of D which do not belong to P constitute an α-ideal (and in fact, a prime α-ideal).

VI. *The status of Partially ordered sets in General Algebra.*

Speaking broadly, Algebra may be said to be concerned with the operational structure of abstract systems. The successive enlargements of the number concept, from the cardinal integers to the real and complex numbers, are effected without changing its basic operational pattern; the further derivates of the number system, like hyper-complex numbers, vectors, matrices, functions etc. do not depart essentially from the operational pattern of the number-system. We may look on 'Modern Algebra' as an attempt to analyse and describe the uniform operational pattern which runs through the number-system and its derivates. According to Modern Algebra, this common

operational pattern may be described by means of the concepts of Group, Ring and Field. It will be noticed that these concepts are independent of the notion of order.

The only other operational pattern which is presented in our fundamental experience, is that which occurs in set-algebra. The operations of set-union and set-intersection are incapable of being described by the concepts of modern algebra; they are doubly distributive and, being tautological, cannot be group-operations. The operational pattern manifested in set-algebra can only be described by the theory of partially ordered sets. It appears from this theory, that, in contrast to the case of real numbers, the order enters in an essential manner in the definition of the operations in set-algebra.

It thus appears that Algebra bifurcates into two coordinate branches, namely the type of operational structure manifested in the number-system and its derivates, and described by Modern Algebra, and the type of operational pattern manifested in set-algebra, and described by the theory of partially ordered sets.

VII. *Applications.*

It is only during the last two or three decades that the theory of partially ordered sets has come to its rightful place as a basic mathematical discipline. The notion of order being fundamental in mathematics, it is natural that it has extended applications to other mathematical theories. Without attempting to go into these in any detail, I shall content myself with mentioning its characteristic applications to Geometry, Logic and Topology, and the work we have been doing in the Mathematics Department of the Madras University. From its inception in Greece, Geometry has always sustained the implicit claim of being a discipline in its own right independent of the number-system. The theory of Partially ordered sets by defining General Projective Geometry as a complemented modular lattice has been able to substantiate this long-standing claim. The way towards this definition has been suggested by the result that the normal subgroups of a group form a modular lattice, and more generally, adopting Emmy Noether's notion of a *group with operators*, the allowable normal subgroups of a group form a modular lattice.

Any set of propositions form a set which is partially ordered by the logical relation of implication, and which is usually a Boolean Algebra or a distributive lattice. The special feature of this system

is that implication has also to be interpreted as an operation in the system. Lattice-theoretic studies on these questions and in particular on Intuitionistic logic have been published by Miss. Pankajam, Mr. K. Chandrasekharan and myself.

We may expect considerable application of partially ordered sets to General Topology, which is concerned with various types of sets. In fact I believe that M. H. Stone's discoveries in the Boolean Algebra and their subsequent extension to distributive lattices were motivated in the first instance by certain problems about the mutual relations of various topologically characterisable kinds of sets, viz. the non-dense sets, the Baire sets, the Borel sets and sets of the first category. Some of us however are interested in a some-what different kind of application of Partially ordered sets to topology. That the different possible topologies on a given set form a partially ordered set (by the ordering relation of stronger and weaker) and a complete lattice, was utilised by Birkhoff in a paper published some twenty years ago. This was also utilised by a student of our department in a thesis on 'The Topology of Resolution-spaces' some years ago. The idea seems to have been neglected after Birkhoff. I believe that considerable light could be thrown by the lattice-study of T_1-topologies on the mutual relationship beween different kinds of topological structure, namely the Hausdroff, the regular, the completely regular, the normal, the completely normal, and the metrical types. Mr. V. S. Krishnan who has done good work on Lattice-Algebra, is engaged in the study of spaces of uniform structure, by the use of this method.

I may mention in closing, the problem in lattice-algebra, of characterising the most general distributive lattice closed for product complements, in which every normal element is simple. The problem has already been met with by M. H. Stone in one of his papers; its solution would not only be of interest in itself, but would also have important applications to logic.

Do You Know?

In 1901 Lindemann published a memoir of seventeen pages purporting to contain the long-sought proof of Fermat's "Last Theorem." The vitiating error being pointed out, Lindemann, undaunted, spent the best part of the next seven years in attempting to patch the unpatchable, and in 1907 published sixtythree pages of alleged proof which were rendered nonsensical by a slip in reasoning near the very beginning.

From the *Annamalainagar Mathematics Exposition.*

Partially Ordered Sets and Projective Geometry

BY

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1. *Historical Introduction*

Geometry, as an independant discipline, may be said to begin with Euclid's "Elements" compiled in the third century B.C., wherein Euclid brought together in a unified and systematic manner the geometrical knowledge of his times. It is famous even to day for its specifically mathematical method of logical deduction from explicitly stated definitions and postulates. His work was carried forward by later Greek geometers, notably Apollonius and Archimedes. Apollonius is remembered for his work on "conic sections" and Archimedes for framing the first postulate regarding what is now recognised as "continuity".

As the next landmark in the history of Geometry we have, in the 16th century, Descartes' discovery of numerical coordinates for points of Euclidean space. This epoch-making discovery made available the powerful methods of Algebra for the solution of geometrical problems. It also marked the beginning of a long and futile controversy between supporters of the algebraic and synthetic methods in Geometry. Descartes, and his contemporary Fermat were the first algebraic geometers; between them they formulated the equations to the plane, the line, the circle, and the conics.

In Euclidean Geometry the "parallel postulate" introduces an asymmetry between "points" and "lines" in the plane, and between "points" and "planes" in space. The removal of this asymmetry and the establishment of what is now called "the principle of duality" mark the distinctive features of Projective Geometry. Projective Geometry may be taken to have acquired an independent status after the publication, in 1822, of the "Traité des propriétés projectives des figures" by the French geometer, Poncelet. In this treatise Poncelet introduces, for the first time, *ideal* space elements like "points, lines and plane at infinity", using which parallelism can be replaced by concurrency at a common *ideal* element. He goes even further in his conception of "the circular points" at

infinity, in terms of which he defines circles as conics passing through these points.

The "principle of duality" makes its first appearance in Poncelet's treatment of reciprocal polars. It was further generalised and extended by Gergonne, Möbius and Plücker, and other geometers of the early 19th century.

Möbius contributed to Projective Geometry what Descartes had done for Euclidean Geometry in his fundamental discovery of homogeneous coordinates for points of a pojective space. Plücker gave a simpler form of homogeneous coordinates in his angular coordinates.

Plücker's claim to fame rests on his remarkable conception of "line coordinates" and of "dimensionality of spaces". By observing the symmetry in 'point'-and 'line' coordinates of the linear form in two variables, he arrived at a simple algebraic explanation of the duality principle.

As one of the greatest synthetic geometers of the age, we may note Steiner who conceived of the different types of '*projective ground forms*' like ranges of points, pencils of lines etc. He observed that a projectivity between two one-dimensional projective ground forms is determined completely by any three pairs of corresponding elements, which proposition was later called "the fundamental theorem of projective geometry". Further he also gave definitions of conics in terms of projective ranges of points, and dually, in terms of projective pencils of lines.

A principal feature in the development of Mathematics in the 19th century was the tendency towards greater abstraction and logical rigor. This trend may be seen exemplified in two of the outstanding achievements of Geometry of the later half of the 19th century; namely, F. Klein's group-theoretic classification of Geometries in his "Erlanger Programm" of 1872, and D. Hilbert's axiomatic treatment of Euclidean Geometry in his "Grundlagen der Geometrie" of 1888.

Following the detailed study of abstract groups by Sophus Lie, Klein observed that to each Geometry could be associated a *group of transformations*, which is characteristic of the Geometry; and the Geometry could be described as "the invariant theory of the group" that is, the study of properties invariant under the transformations of the group. By comparing the associated groups, Klein was able

to classify the different Geometries. Thus Euclidean and affine Geometries are seen to be subsumed under Projective Geometry, since the groups associated with these are subgroups of the "projective group".

Hilbert's "Grundlagen" is remarkable for the thoroughness with which it exhibits Euclidean Geometry as a "hypothetico-deductive theory."* The discovery of non-Euclidean Geometries by Lobatchewsky, Bolyai and later by Riemann had renewed the interest in the axiomatic basis for Euclidean Geometry. Pasch, and later Peano, gave axiomatic treatments of Euclidean Geometry. But it was Hilbert's work which furnished a really simple and satisfactory axiomatic approach to Euclidean Geometry. The salient features of his "Grundlagen" are (1) the simplicity and completeness of his system of axioms, among which occur some that had been used by Euclid without explicit mention as axioms, (2) the classification of these axioms into related groups, and (3) the study of the mutual logical dependance of these groups of axioms. To establish the independence of certain of them, Hilbert has constructed new Geometries, namely a non-Desarguesian Plane Geometry and a non-Archimedean Geometry.

An axiomatic treatment of Projective Geometry on parallel lines emerged into final form from the work of many 19th century geometers starting with Von Staudt who tried to develop Projective Geometry without using the metric implicit in the numerical cross-ratio, by constructing his "algebra of throws". At the present day we have completely axiomatic treatments of the subject. We shall now proceed to consider the salient features in the axiomatic approach to Projective Geometry.

2. *The axioms of Projective Geometry*

Classical Projective Geometry may be called, following Veblen and Young, "Real Projective Geometry". The complete set of axioms for this deductive theory fall naturally into three groups, and by stopping at intermediate stages we get two other deductive theories of a more general nature; these are "General Projective Geometry" where we consider the first group of axioms alone, and "Projective Geometry over an abstract field", where we consider the first two groups of axioms.

The "elements" we have to deal with in Projective Geometry (viz., points, lines, planes, etc.) are all subsets of a certain basic

* We may note here the importance of this work for the later formulation by Hilbert of the "formalistic" view of Mathematics", for the "Grundlagen" may be taken to form the starting point for "Formalism".

set S. With each of these elements (A) is associated an integral "*dimension*" (dim A). The null-set considered as a subset of S is the only element of dimension (-1). The one-element subsets of S form the elements of dimension 0, and are also called '*the points*'*. The 1, 2, . . . n dimensional elements are also called, in order, *lines*, planes, . . . n-spaces.

The first group of axioms deal with this "dimension" and an "incidence relation", *on*, which is just the natural partial-ordering relation of "set-inclusion" among the elements considered as subsets of S. These axioms are:

A1: Any two distinct points A, B are on one, and only one line (A, B).

A2: There are at least three points on each line.

A3: If A, B, C are three distinct points not all on one line, and D, E ($D \neq E$) are points on (A, C), (B, C) respectively, then there is a point F on both (A, B) and (D, E).

A4: In any $(k+1)$ dimensional element A we can find a k dimensional element B and a point P, not in B, such that every element containing B and P contains A, and the points in A are all the points on lines joining P with the different points in B.

A5: There is an element I of dimension n, and there are no elements of dimension $> n$ or < -1.

These axioms determine an "n-dimensional general projective geometry", in which many of the familiar results relating to the dimension and incidence hold good. We shall note here only some and refer the reader to Veblen and Young's book for a further treatment.

(*a*) Given any two elements A, B two other elements can be constructed uniquely, namely their '*join*' A+B, and their '*meet*' A·B. The "join" consists of the points on the lines joining the points in A with those in B, while the "meet" consists of the points common to A and B. The dimensions of these four elements satisfy the 'dimensional law': dim (A) + dim (B) = dim (A + B) + dim (A·B).

(*b*) Any k-dimensional element is the join of any $(k+1)$ '*independent*' points in it – i.e., $(k+1)$ points of which no $(l+1)$, for $l \leqslant k$ lie on an element of dimension $< l$.

*By identifying the single elements and the one-element subsets, we may also regard the "elements" of our Geometry to consist of sets of "points".

The second group of axioms consist of the two propositions called *Desargues' theorem* and *Pascal's theorem* for a pair of intersecting lines.* Desargues' theorem states that if two triangles are perspective from a point they are coaxial, and vice-versa. It is a remarkable result due to Hilbert that this proposition is a consequence of the axioms, A1 to A5 if $n \geqslant 3$, while it is independent of these axioms if $n=2$, as he proves by constructing a non-Desarguesian plane geometry. Pascal's theorem says that if a simple hexagram is inscribed in a pair of intersecting lines, the three pairs of its opposite sides intersect in three collinear points.

If Desargues' theorem is assumed in addition to axioms A1 - A5, then it is possible to determine geometrically two binary operations + and · for the points on any line l, with respect to three arbitrary chosen basic points P_0, P_1, P_α on l, such that under these operations the points of l, other than P_α, form an abstract field F. This field has been shown, by Hilbert, to be commutative if, and only of, Pascal's theorem is also assumed. On assuming both these theorems we can associate with the points of the space non-homogenous or homogeneous coordinates from this field F (whose structure, it may be remarked, does not depend on the choice of the line l or of the points P_0, P_1, P_α on l).

The final group of axioms is given different forms by different writers; but these usually deal with notions of "*order*" and "*continuity*" among the points of a line. These axioms are so framed that the associated field F now becomes isomorphic with the field of real numbers so that we get "Real Projective Geometry" with real number coordinates for the points. But it is the general Projective Geometry defined by the axioms A1 - A5 which is of interest to us here, on account of its connection with Partially ordered sets. We shall examine this connection in the following section.

3. *General Projective Geometry as a complemented, modular lattice of finite dimension.*

Set inclusion partially orders any family of sets. So, since the "elements" of General Projective Geometry of n-dimensions defined by the axioms A1 - A5, are all subsets of a basic set S (or, alternatively of the set of all "points") these "elements" form a partially ordered set, $\overline{P}$, with set inclusion (or "the incidence

* These two propositions can be proved as theorems in classical or *real* projective geometry wherein points have real number homogeneous coordinates.

relation") as ordering relation. From the definitions of "the join" and "the meet" of two elements, given earlier, it may be deduced that these form, respectively, the *lattice-sum* and the *lattice-product* of the two elements. So $\overline{P}$ is a lattice. To obtain a complete characterisation of this lattice we shall define "finite dimensional lattices" and a notion of "dimension" for the elements of such a lattice.

In a partially ordered set, P, we say that an element a '*covers*' another b if $a > b$, and no other element c of P lies between a and b, so that $a > c > b$ implies $a = c$ or $c = b$. Next we define a sequence $x_0, x_1, \ldots x_n$ of elements of P to be a "descending chain" of length n joining a with b if $a = x_0$, $b = x_n$ and x_{i-1} covers x_i for $i = 1, \ldots n$. The partially ordered set P is said to be "finite dimensional" if there is a finite upper bound to the lengths of all descending chains joining pairs of elements in it.

If a lattice L is of finite dimensions, then it contains a (unique) *zero* '0' and a (unique) *one* '1' satisfying the conditions $0 <$ every element of $L < 1$. And in such a lattice we can associate to each element a a lattice-dimension '$d(a)$', namely the maximum of the lengths of descending chains joining a with 0. The number $d(1)$ is also called the dimension of the lattice L, and denoted by $d(L)$. Elements of dimension 1, that is the elements that cover 0, are called 'atoms'. It may be easily verified that the dimension is 'monotonic'; that is, $a \geqslant b$ implies $d(a) \geqslant d(b)$, while if $a > b$ and $a \neq b$, then $d(a) \geqslant d(b) + 1$.

We now observe that the lattice $\overline{P}$ is finite dimensional. In fact, it can be established without difficulty, that the lattice dimension of any element of $\overline{P}$ is 1 greater than its geometrical dimension. Hence, as I is the whole space, and it contains all other elements, $d(\overline{P}) = d(I) = n+1$. Also the dimensional law deduced earlier for the geometrical dimension obviously implies a similar law for the lattice-dimension. So, for any a, b from $\overline{P}$, we have

$$d(a) + d(b) = d(a+b) + d(a \cdot b).$$

By using this law, we shall now show that the lattice $\overline{P}$ is "*modular*". For, if a, b, c are any three elements of $\overline{P}$ and $a > b$, then $a \cdot (b+c) > b$ and $> a \cdot c$. So $a \cdot (b+c) > b + a \cdot c$. On the other hand

$$d\,[a \cdot (b+c)] = d(a) + d(b+c) - d\,(\overline{a+b+c}), \text{ since } d(a) + d(b+c) = d(a+b+c) + d\,[a \cdot (b+c)];$$

$$= d(a) + d(b+c) - d(a+c), \text{ as } a+b = a \text{ when } a > b;$$

$$= d(a \cdot c) - d(c) + d(b+c), \text{ since } d(a) + d(c) = d(a+c) + d(a \cdot c);$$

$= d(a \cdot c) + d(b) - d(b \cdot c)$, since $d(b) + d(c) = d(b+c) + d(b \cdot c)$;

$= d(a \cdot c) + d(b) - d(\overline{b \cdot a \cdot c})$, since $b \cdot a = b$, when $a > b$.

$= d(b + a \cdot c)$, since $d(b) + d(a \cdot c) = d(b \cdot a \cdot c) + d(b + a \cdot c)$

Now, since $a(b+c) > b + a \cdot c$, $a \cdot (b+c) \neq b + a \cdot c$ would imply $d\,[a(b+c)] \geqslant d(b + a \cdot c) + 1$, contradicting the above equality. So $a \cdot (b+c)$ must be equal to $b + a \cdot c$, which proves the 'modularity' of $\overline{\mathrm{P}}$.

This lattice $\overline{\mathrm{P}}$ is also "complemented"; that is, each element, a, of $\overline{\mathrm{P}}$ has at least one "*complement*", a', such that $a \cdot a' = 0$ and $a + a' = 1$. For if $d(a) = k$, we can find $(k+1)$ independent points $a_1, \ldots a_{k-1}$ in a, and, in I, $(n+1)$ independent points containing these $(k+1)$ points, say $a_1, \ldots a_{n+1}$ such that $a = \sum_{i=1}^{k+1} a_i$ and $\mathrm{I} = \sum_{i=1}^{n+1} a_i$. Then it can be easily verified that $a' = \sum_{i=k+2}^{i=n+1} a_i$ is a 'complement' of a.

Finally Birkhoff has proved that $\overline{\mathrm{P}}$ is '*simple*' that is, it cannot be factorised as the '*product*' of other (non-trivial) partially ordered sets. He deduces this from a general result he establishes, namely that a complemented modular lattice of finite dimensions is 'simple' if, and only if, in it the sum of any two distinct 'atoms' contains a third 'atom'. This condition is implied for $\overline{\mathrm{P}}$ by axiom A_2, since the points form the atoms of $\overline{\mathrm{P}}$, and the join of two distinct points is the line containing them.

Conversely also, Birkhoff has shown that a complemented, modular lattice of dimension $(n+1)$ can be exhibited as a general Projective Geometry of dimension n, if the lattice is 'simple'—by taking the atoms to be 'points', 2 dimensional elements to be 'lines', and generally $(k+1)$ dimensional elements to be 'k-spaces'.

This lattice structure of the General Projective Geometry affords an immediate proof of the duality principle for this Geometry. For since all the restrictions on $\overline{\mathrm{P}}$ are self dual, the *dual* of $\overline{\mathrm{P}}$-obtained by replacing the ordering relation $<$ by its inverse $>$ is also a simple, complemented, modular lattice of dimension $(n+1)$.

References

(1) Encyklopadie der Mathematischen Wissenschaften, Band III: 1-2.

(2) E. T. BELL. Development of Mathematics.

(3) VEBLEN AND YOUNG—"Projective Geometry" in two volumes.

(4) A. N. WHITEHEAD. "Axioms of Projective Geometry" (Cambridge tract).

(5) K. MENGER—"New foundations of projective and affine geometry", *Annals of Math.*, Vol. 37 (1936).

(6) G. BIRKHOFF—"Lattice Theory", American Mathematical Soc., Colloquium Publications, Vol. XXV.

Partially Ordered Sets and Symbolic Logic

BY

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1. Symbolic logic is the investigation, by the symbolic method of the formal structure of deductive reasoning. While the use of symbols to denote classes characterized logic from the beginning (as, for instance, in the use of letters to represent the terms of a syllogism) the employment of symbols to denote operations was not foreseen until the time of Leibnitz. The discovery of a kind of Algebra which will bring out into relief the structure of logic, and the achievement of a workable calculus for the application of operations of mathematical type to logic are really the work of George Boole.

2. Leibnitz was the first serious student of Symbolic Logic. He imagined the possibility of a 'universal calculus of reasoning', and attempted to realize it by symbolizing logical operations. Thus, he stated the principal properties of logical addition, multiplication, negation, the null-class and class-inclusion; and he noted the similarity of certain properties of inclusion for classes and implication for propositions. Leibnitz's efforts were followed up by Lambert, Holland, Hamilton and De-Morgan. While they did not succeed in setting forth any concrete calculus, their work, no doubt, paved the way for Boole.

3. Boole is really the founder of modern Symbolic Logic, "the man who gave Leibnitz's dream its first substance."(1) His algebra, presented in 1847, forms the basis of the whole development since. It can be interpreted as a calculus of classes or a calculus of propositions. I shall briefly outline Boole's system.(2)

The three main ideas of Boole are (i) the operation of election and the introduction of elective symbols (ii) the laws of thought which can be expressed in terms of operations upon these symbols (iii) the identification of these laws with the laws of a two-element 'Boolean Algebra' consisting of 0 and 1.

The elective symbol x is the result of selecting all the x's in the universe. x may be thought of as a predicate determining a certain class. The 'operation' of election thus becomes an operation of class-formation.

This operation of election is analogous to algebraic multiplication. For, if we first select the x's and from them select the y's, these two operations in succession lead to the class of things common to x and y. This new class may be represented by $x \cdot y$.

It is obvious that $xy = yx$. Also, if $x = y$, then $zx = zy$. We have, besides, $xx = x$, which is the peculiar law that distinguishes this algebra from ordinary algebra.

Boole considers next the operation of 'aggregation' standing for 'either . . or'. The class of objects which are either x's or y's is represented by $x + y$, where x and y are supposed to be disjoint. Then, obviously, $x + y = y + x$. Also $z(x + y) = zx + zy$.

Next, he takes up the operation of 'exception'. The class of things which are x's and not y's is represented by $x - y$. Then, we can verify that $z(x - y) = zx - zy$. He assumes $-y + x = x - y$ as a convention.

1 is taken to represent everything; 0 represents nothing. These meanings accord with $1 \cdot x = x$ and $0 \cdot x = 0$.

The 'negation' of any class x is $1 - x$. With this definition, we have $x + (1 - x) = x + 1 - x = 1$; which means that everything is either x or not x. Also $x(1 - x) = x - x^2 = x - x = 0$, which means that nothing is both x and not x. Thus the introduction of 0 and 1 leads to the law of the Excluded Middle and of Contradiction.

4. The distinctive law of Boole's system is $xx = x$. This holds of no other numbers than 0 and 1. We can therefore suppose Boole's system as a numerical algebra of 0 and 1 independent of any logical interpretation. This algebra is what we now call Boolean Algebra. It is the basis of modern Symbolic Logic. The laws of logic can be obtained as a result of performing first the operations

of this numerical algebra, and then restoring to the results their logical meaning.

In restoring that logical interpretation, there were difficulties with regard to Boole's operation of addition which was restricted to disjoint classes. Thus $1+x$, $x+x$ lacked a definite meaning. This inconvenience was got over by his successors like Jevons, Venn, Peirce and Schroder. They interpreted $x+y$ to mean 'x or y or both'. Thus $x+x=x$, $1+x=1$. With the new meaning, the negative of xy became $(1-x)+(1-y)$; or, in current notation, $x'+y'$. The operations of subtraction and division were eliminated, and a new relation of 'inclusion' of a in b, symbolized by $a<b$, was introduced so as to mean that every 'a' is a 'b'. By the incorporation of these ideas, Schroder consolidated the algebra of Boole, and gave it its present form.

5. Besides interpreting his algebra as a calculus of predicates, Boole pointed out an important application. He considered propositions in which the predicate is given but the subject is a variable. Any term a could be taken to represent the class of cases in which the predicate a is true. Then a' will represent the class of cases in which the predicate a is false. ab will represent the class of cases in which both the predicates a and b are true; $a+b$ will be the class of cases in which either the predicate a or the predicate b is true. In present day usage, this interpretation can be said to apply to propositional functions, as distinct from propositions. A proposition is either always true, or always false; whereas a propositional function may reduce to a true proposition for one assigned value of the variable, and reduce to a false proposition for another value. In Boole's Algebra, $a=1$ means 'a' is always true, and '$a=0$' means a is never true. In the case of propositions we have the property: if $a\neq 0$ then $a=1$. This property is not true of propositional functions.

Boole's work consisted in constructing a calculus of predicates and extending it to construct a calculus of propositional functions. His followers, Peirce and Schroder have developed the calculus of propositions and the calculus of propositional functions separately.

6. It was at this time (1895) that Peano and his colleagues began their attempts to exhibit every branch of Mathematics in such a way as to be derivable by rigorous deduction from a specified number of assumptions. The symbolization of logic and its presentation in an accurate and handy form by Boole and Schroder offered

a helping hand. In fact, the investigations of Peano showed that logic itself had to be expanded in order to facilitate the symbolization of mathematical concepts. Frege had worked on the same lines much earlier, but his results passed unnoticed until a later date (1901).

7. The 'Principia Mathematica' of Russell and Whitehead formed the next noteworthy contribution to the subject. It had the same object in view as Peano's 'Formulaire', namely, the exhibition of every branch of Mathematics in a deductive fashion; but its achievement was far more substantial. Whereas Peano used for the development of arithmetic several extra-logical postulates, Russell and Whitehead demonstrated that these extra-logical postulates could be dispensed with, and that arithmetic could be derived from only the primitive assumptions of logic. For the sake of clarity, they developed logic itself, in a deductive fashion, from a small number of undefined ideas and a few postulates involving these ideas. It was proved that these primitive ideas and postulates for logic were the only assumptions required for the whole of Mathematics. The propositions of Mathematics could be deduced as logical consequences of the primitive assumptions of logic. Thus Mathematics could be viewed as a branch of logic, which consisted of all propositions of the form $f(x, y, z, \ldots)$ implies $\phi(x, y, z, \ldots)$ where $x, y, z \ldots$ vary over classes and the only other notions involved in f, ϕ are logical constants. The Principia is a landmark in the history of mathematics. Though its treatment has since been further simplified, its importance has not been lessened.

8. Russell's work revealed not only that the structure of Mathematics was based on logic, but also the fact that logic itself possessed an abstract structure in as much as it could be subjected to deductive treatment. This suggested to Hilbert the idea of separating Mathematics from logic and formulating the concept of an [8] abstract deductive system underlying both of them.

Consider any abstract set of elements K, and any abstract set L of relations and operations for which the set K is closed. Consider the set P of all assertible propositions about the elements of K as related by the members of L. "Where these propositions are organised in a system such that all of them are derivable from some proper part of them in accordance with the principles of deductive logic, the system K, L, P constitutes what is called a deductive system. The Boole-Schröder algebra is an example of such a system. An abstract deductive system may have more concrete interpretations than one. For instance, the Boole-Schröder Algebra can be

interpreted as a calculus of classes, or a calculus of propositions, or a calculus of areas in a plane.

To exhibit logic itself as a deductive system, Hilbert set up a higher level of abstraction.[2] Consider any abstract set of elements K, and any abstract set L of relations and operations for which K is closed, and consider the set P of all *combinations* of the elements of K and L which can be *derived* from given combinations by the application of specific rules of procedure. Then K, L, P constitute a deductive system of the second level. It is important to notice that, at this level, we are concerned with not even propositions, but some combinations of symbols, and we *derive* new combinations of symbols from the given ones using for such derivation only given rules of procedure. This derivation, unlike proof, does not require the exercise of the faculty of thought, and is governed only by the specified rules, irrespective of the nature of the elements under consideration. Hilbert does not consider proofs as part of the deductive system. The proofs of the formulae of a deductive system and the rules of inference employed are separated so as to constitute what is called a meta-system.

9. Thus, in contrast with the logistic view of Mathematics advanced by Russell in his assertion that logic is the foundation of Mathematics, Hilbert claims that logic can be exhibited as a branch of Mathematics if only we define Mathematics as the set of all abstract deductive systems as distinct from meta-systems. Powerful opposition to these theories has been offered by the Intuitionists led by Brouwer. The singular features of intuitionistic logic and the revolutionary changes it entails in Mathematics are worth studying for their intrinsic interest.[6] Such a study will, however, be greatly facilitated by the adoption of modern lattice-theoretic methods in the analysis of logic.

I shall therefore now examine how the idea of partially ordered sets makes its entry into this study, and how it provides us with an advantageous view-point.

10. Consider the set P of all elementary propositions i.e. propositions considered as a whole without separation of subject and predicate. Take the relation of implication as fundamental in the set. Thus, 'a implies b' is denoted by $a<b$. We assume that this relation is reflexive and transitive. Further, if $a<b$ and $b<a$, we say that $a=b$. The relation $<$ is some sort of 'deducibility' possessing no properties other than those specifically assumed. These definitions make P a *partially ordered set.*

A partially ordered set K is a lattice if the following postulates hold. (i) For any two elements a, b of K there exists an element c of K called the sum $a+b$ such that $c>a$, $c>b$, and $x>a$, $x>b$ leads to $x>c$. (ii) for any two elements a, b of K there exists an element d called the product ab such that $d<a$, $d<b$, and $a>x$, $b>x$ lead to $d>x$.

The lattice K is distributive if $a(b+c)=ab+ac$ whatever be the elements a, b, c of K.

In the partially ordered set of propositions that we are considering, we can take 'a or b' to be the sum, and 'a and b' to be the product respectively, of the propositions. Then the above two postulates hold, so that P becomes a lattice. It can be easily seen that P is, in fact, distributive.

This distributive lattice contains units. Any self-contradictory proposition forms the zero, and any logically necessary proposition forms the one. These units are unique in the sense of equivalence.

11. We shall now introduce the unary operation of negation denoted by ($'$ or $\sim$) into the distributive propositional lattice P. In so doing, we fix the meaning of that operation by the following requirements.

1. Law of contradiction $aa'=0$.
2. If $ax=0$, then $x<a'$
3. Law of excluded middle $a+a'=1$
4. Law of double negation $a''=a$.

Thus, $aa'=0$, $a+a'=1$, $a''=a$, $0'=1$, $1'=0$. These render P into a complemented distributive lattice or a Boolean Algebra B where Boolean complementation stands for negation.

12. We have seen that the relation of lattice-order is identifiable with the *relation* of implication. It is however interesting to start anew with the Boolean Algebra B, and introduce in it the *operation* of implication, so that if a and b are two propositions, $a \supset b$ is another proposition.

$$\text{Define } a \supset b \ \cdot\!=\!\cdot \ a'+b.$$

With this definition we can verify several properties that are commonly associated with implication, such as reflexivity and transitivity.

We also define $a \equiv b \ : = : \ a \supset b \cdot b \supset a$.

The connection between $>$ and $\supset$ is established by the following :—

$$a<b \text{ if and only if } a\supset b=1.$$

For, if $a<b$, then $a'>b'$ so that $a'+b>b'+b=1$ i.e. $a\supset b=1$.

If $a\supset b=1$, then $a'+b=1$ or $ab'=0$. But $ab+ab'=a\,(b+b')=a$

Therefore $ab=a$ or $a<b$.

If, in addition, we choose to say that $a\equiv(a=1)$ and $a'\equiv(a=0)$ then, the Boolean Algebra reduces to the two elements 0 and 1 under equivalence. If $a=0$, we say, a is false, and if $a=1$ we say, a is true. Thus, false and self-contradictory propositions are the same, while true and necessary propositions are the same. This system of logic is called Material Implication. It is developed in Russell's 'Principia'.

13. Some of the revealing features of Material Implication are :—

(i) $a\supset b$ holds in every case except when a is true and b is false. Thus 'grass is green' materially implies 'Rose is red', because both the propositions are true, though we may not be able to deduce logically the second proposition from the first.

(ii) No system of postulates can be at once consistent and independent. P is consistent with Q means P does not imply Q'. Q is independent of P means P does not imply Q. Now, $P\supset Q$ or $P\supset Q'$ is a provable formula.

(iii) P is logically necessary ($\sim P\supset P$), and P is true, have the same meaning since $P\equiv(P=1)$.

14. The calculus of strict implication due to Lewis [10] is obtained by defining the operation of strict implication $\prec$ in the Boolean Algebra of Propositions with which we started. The operation $\prec$ is defined through another unary operation $\Diamond P$ called the 'possibility of P'.

$$\Diamond p=1 \text{ if and only if } p\neq 0$$

$$\Diamond p=0 \text{ if and only if } p=0$$

Define $\qquad p\prec q \cdot = \cdot \sim \Diamond\ (p\sim q).$

We now see that $p\prec q=1$ if and only if $p<q$.

The peculiarities of this system are :—

(i) We cannot assert that

$$\sim(p\sim q) : \prec : (p\prec q). \text{ Thus } \sim(p\sim q)\neq p\prec q.$$

(ii) We cannot assert that $p \equiv (p=1)$ since $p ⥽ (p=1)$ does not hold. Thus '*p* is true' differs from '*p* is necessary'; and '*p* is false' differs from '*p* is absurd' Instead of the two modes of propositions of Material Implication, we have now four modes.

(iii) $p ⥽ q$ or $p ⥽ q'$ does not hold. There exist propositions which neither imply a given proposition nor its contradiction.

(iv) It follows from (iii) that the Boolean Algebra has more than two elements.

(v) In any propositional expression, the assertive implication $\supset$ can be replaced by the assertive implication $⥽$ since $p ⥽ q=1$ and $p \supset q=1$ are the same as $p<q$. But the subordinate implication $\supset$ cannot be replaced by $⥽$.

(vi) In Mathematics, whose prime object is to state that certain primitive propositions *imply* certain conclusions, the implication that occurs is *strict*.

15. So far, we have been concerned only with the differences in the operation and interpretation of 'implication', fixing the other operations like negation and disjunction. The logic of Intuitionists formalized by Heyting is somewhat more interesting. For, negation no longer satisfies the four requirements postulated above (in § 11). It rejects the law of the excluded middle, and consequently the law of double negation as well.

Garrett Birkoff[4] has given a lattice-theoretic treatment of this logic as a 'Brouwerian' lattice. S. Pankajam[12] has also worked on the same lines. Their conclusion is that if we consider a distributive lattice of propositions P which is closed for product—complements and relative product-complements, then the product complement corresponds to the intuitionistic negation and the relative product complement corresponds to the *operation* of implication. The set of simple elements of the lattice, namely, those which satisfy the law of the excluded middle, constitutes a Boolean Algebra, which is in conformity with ordinary logic.

The logical interpretation of these results is interesting. I have explained it at some length in one of my published papers.[5] Since $a''<a$ does not hold in this logic, the proofs that we adopt must be *constructive* and not merely based on *reductio ad absurdum*. This constructive demand has its effect on the meaning of 'or'. $a+b$ does not mean merely 'if not '*a*, then *b*,' but 'a specifiable one of *a*

and *b*.' The meaning of this strong disjunction has been very well clarified by Dr. R. Vaidyanathaswamy.[14] I have explained elsewhere the theory of linear order [6] according to Intuitionists, which will adequately illustrate intuitionistic reasoning.

16. It should be noted that though the Principle of the Excluded Middle fails to be universally true in Heyting's logic, it is not absurd. Thus, as in Strict Implication, a difference is made between $a \neq 1$ and $a = 0$. However, Heyting's logic is a distributive lattice while Strict Implication is a Boolean Algebra. In either case, there exist more elements than two. The principle of Excluded Middle holds in Strict Implication, and with that the Law of Double Negation. This point is confused in Birkoff's book.[4]

17. That the logic of Strict Implication gives rise to four modes of propositions instead of the classical two, was noticed earlier. In fact, Lukasiewicz and Tarski [10] have constructed logics containing n distinct truth-values. Before we proceed to examine their view-point, it is interesting to recall the calculus of propositions as developed in the Principia.

The system developed therein can be interpreted as a truth-value system. To every proposition P is attributed a truth-value $t\ (p)$ which is 0 or 1 according as it is false or true. And the following definitions are made: $t(p+q) = \text{Max}\ [t(p), t(q)]$; $t(p \cdot q) = \text{Min}\ [t(p), t(q)]$; $t(p') = 1 - t(p)$; $t(p \supset q) = \text{Max}\ [1 - t(p), t(q)]$. The laws of the Propositional calculus are those propositions which have the truth-value 1 whatever be the truth-values of the variables, occurring in them.

It is this idea that Lukasienicz and Taski have generalized. They have constructed a logic on the $n+1$ values $0, \frac{1}{n}, \frac{2}{n}, \cdots \frac{n-1}{n}, 1.$ We can picture it formally as follows.

Let us start with a set of elementary propositions P which is closed for an undefined operation $\supset$. Let us assign to every proposition p a truth-value $t(p)$ from out of the $n+1$ given values. Let us further define

Df.: $t(p \supset q) = 1$ if and only if $t(p) \leqslant t(q)$.
$t(p \supset q) = 1 - t(p) + t(q)$ if $t(p) > t(q)$.

Df. $p < q$ if and only if $t(p \supset q) = 1$.

(i) The relation $<$ is reflexive since $t(p \supset p) = 1$

(ii) It is transitive. For, suppose $p < q$ and $q < r$. Then $t(p) \leqslant t(q) \leqslant t(r)$ which means that $t(p) \leqslant t(r)$ and so $p < r$.

(iii) If $p<q$ and $q<p$ then we say that $p\equiv q$. In other words, if $t(p)=t(q)$, then $p\equiv q$.

Thus the set of propositions P is partially ordered by this relation $<$.

Now, define the sum of p and q thus:

Df: $t(p+q)=\text{Max}\ [t(p),\ t(q)]$

We easily see that, with this definition, + satifies the conditions for a lattice-sum, namely, $p+q>p$, $p+q>q$ and $r>p$, $r>q$ imply $r>p+q$.

Similarly, define $t(p \cdot q)=\text{Min}\ [t(p),\ t(q)]$.

Thus P becomes a *lattice.* This lattice is, in fact, distributive since $p(r+s)\equiv pr+ps$.

We define negation thus: $t(p')=1-t(p)$. We then prove that $p<q$ leads to $p'>q'$ and that $p''\equiv p$.

"The actual meanings that have to be attached to these operations must be found from considerations of probability."

It is surprising that $[a.\ a\supset b]\supset b=1$ does not hold, as for example, when $t(a)=\frac{2}{3}$, $t(b)=\frac{1}{3}$ in the 3 valued logic.

All provable formulæ of this logic hold good in the two-valued case, but not conversely. Unlike Heyting's logic, here the principle of excluded middle, understood as $a+a'=1$, is rejected, but the law of double negation is retained.

18. The idea of truth-values has been used by Hilbert (8) to test the consistency and independence of the system of postulates for logic. If x,y,z...are the variables occurring in the postulates, we assign one of the truth-values 0, 1 to each of them, and assume that $0'=1$, $1'=0$. Then the system is consistent if and only if all the postulates have the truth value 1 or all of them have 0.

To prove that the postulates are independent, or in other words, that no single postulate P could be derived from the rest, such a system of truth values is invented that the truth-value of the postulate P is different from those which the other postulates have in common.

19. A survey of the various new algebras of logic points to the possibility of starting with any enumerable set of propositions P which form a distributive lattice, assigning one truth-value to each,

and studying the correspondence between P and T (P). It follows without difficulty that T (P) is also a distributive lattice. But suitable definitions for negation and implication have to be found such that all the operations are preserved by the homomorphism from P to T (P). The matrix method of Tarski (5) seems to confirm this possibility, but a precise solution constitutes a problem for research.

References

1. BELL, E.T : Development of Mathematics
2. BENNETT & BAYLISS : Formal Logic
3. BOOLE, G. : Mathematical Analysis of Logic
4. BIRKOFF, G. : Lattice-Theory.
5. CHANDRASEKHARAN, K. : The Logic of Intuitionistic Mathematics, *Math. Student* (1941)
6. CHANDRASEKHARAN, K. : Intuitionistic Theory of Linear Order *Math. Student* (1942)
7. FRINK, O : New Algebras of Logic, *American Math. Monthly*, 1938.
8. HILBERT & ACKERMANN : Grundzugen der Theoretichen Logik.
9. „ & BERNAYS : Grundlagen der Mathematik.
10. LEWIS & LANGFORD : Symbolic Logic.
11. LANGER, SUSAN : Symbolic Logic.
12. PANKAJAM, S : The Formal Structure of the Propositional Calculus I & II *Jour. Ind. Math. Soc.* (1942).
13. RUSSELL & WHITEHEAD : Principia Mathematica
14. VAIDYANATHASWAMY, R : On Disjunction in Intuitionistic Logic *Proc. Ind. Acad. Sci.* (Feb. 1943)
15. TARSKI, A. : Der Aussagenkalkul und die Topologie. *Fund. Mathematica*, (1938) Vol. 31.

Group Operation in Certain Distributive Lattices

BY

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§ 1. The question of the existence of a group operation in Distributive Lattices first arose in the consideration of Boolean Algebras. Take first the simple case of the algebra S of all sub-sets of a set in which $+$, $\times$, $'$ denote set-sum, set-product and set-complement respectively. If in this S, the operation $\oplus$ be defined as $a \oplus b = ab' + a'b$ (where a, b are any two of the sub-sets), it is easily established that all the above sub-sets of a set S, form a group with respect to the operation $\oplus$. For, from the definition, it is seen that $b' \oplus b = 1$ and also $a \oplus b = a$ if and only if $b = 0$. Hence, the set of elements include a null-set $= 0$. Again, $a \oplus a = 0$ so that each element is its own inverse. Finally, $\oplus$ is associative and commutative. Hence, the sub-sets of the set S, form a group of order 2 with $\oplus$ as group operation.

Next, a general Boolean Algebra B being formally similar to the algebra of all sub-sets of a set, it can similarly be seen that if a, b be any two elements and a', b' their boolean complements and if further, $a \oplus b$ be defined as $ab' + a'b$, then * the operation $\oplus$ will give a group operation in B with identity element 0 and each element of order 2.

The most general group operation was first explicitly determined by B. A. Bernstein† ; later, M. H. Stone § based his study of Boolean Algebra on this group operation using the same to convert the former into a classical ring. Recently Dr. Vaidyanathaswamy †† has shown that all the group operatives of Bernstein can be derived from any one of them by a process known as 'paraphrasing' (for instance, the group operation $a \oplus b$ as above).

But so far, the existence of such a group operation in general distributive lattices has not been considered and also, the precise

* See my paper—"On Finite Boolean Algebra" *The Math. Student* Vol. 6.

† B. A. Bernstein:—'Operations with respect to which the elements of a Boolean Algebra form a Group.' *Trans of Amer. Math. Soc.* 26 (1924) pp 171—174.

§ M. H. Stone:—'Theory of Representations for Boolean Algebras'. ibid. 40 (1936).

†† R. Vaidyanathaswamy:—'Group operations of the Boolean Algebra' J. I. M. S. 1937. Vol. II. No. 6.

For further details on this, refer to my paper 'On Finite Boolean Algebra' cited earlier.

conditions under which such a lattice with O, 1 might admit a group operation has not been investigated. It is even doubtful whether the mere existence of a dual automorphism with the properties $(a+b)' = a'b'$, $(ab)' = a' + b'$ holding among the elements will be sufficient to ensure the possibility of such a group operation. It can however be shown that in a specially simple type of distributive lattices where dual automorphism always exists, a group operation can be defined in terms of lattice order and the dual automorph of the chain and its sub-chains. Further, it can also be deduced that a group operation will also exist in the direct product of such finite chain lattices (called Quasi Boolean Algebras).

The object of this paper is to establish the above two results.

§ 2. *The chain Lattice.*

A partially ordered set is a chain if it is simply ordered i. e. if for any two distinct elements a, b, $a>b$ or $b>a$ holds. Clearly, a chain is always a lattice; for if a, b be any two distinct elements, the lattice-sum* $a+b$ and lattice-product ab is the 'greater' and 'lesser' of the elements a, b respectively. Also, since the distributive law holds in the chain, it is further a distributive lattice.

Consider now, a finite chain C of $N+1$ elements say,—a_0, a_1, a_2, $\cdot \cdot \cdot \cdot$, a_{N-1}, a_n where $a_M < a_n$ if $M<N$. Clearly, there exists in the chain a dual automorphism which carries any element a_M into a_{N-M}; this is a bi-univocal operation and it is a dual automorphism as it simply inverts the order of elements in the chain. Hence, denoting the dual automorphism by a stroke ($'$), it follows that,—$(a+b)' = a'.\ b'$; $(ab)' = a' + b'$ and $a \cdot > \cdot b\ :\supset:\ a' \cdot < \cdot\ b'$. Since every sub-set of a finite chain is also a finite chain, we can also consider the dual automorphism of sub-sets of a chain. Here we are concerned with two types of sub-sets,—namely the upper and lower segments of an element of the chain, the former being the sub-set of *all* elements $\geqslant a$ while the latter is the sub-set of *all* elements $< a$.

Definition 1. The dual automorph of b relative to a is the automorph of b in the upper or lower segment of a according as b lies in the upper or lower segment.

This may be denoted by the quotient b/a.

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* For a general theory of lattices and in particular Distributive Lattices see H.M. Mac Neille.—Theory of Partially Ordered Sets, *Trans of Amer. Math. Soc.* 42 (1937), pp. 416—60.

On account of the unsymmetry of our definition of the upper and lower segments, the quotient b/a will not be an invariant for the dual automorphism of the chain, i.e. the relation $(b/a)'=b'/a'$ will not hold always.

THEOREM 1. *$(b/a)'$ is the immediate successor or predecessor of b'/a' according as $b<a$ or $b>a$ respectively.*

Proof. Case 1. let $p>q$ so that $a_p>a_q$.

Then, a_p/a_q is the element $a_{N-(p-q)}$ whereas $a'_p/a_q'=a_{N-p}/a_{N-q}$ is equivalent to the dual automorph of a_{N-p} in the lower segment of a_{N-q} (the highest element of the lower segment here being $a_{N-(q-1)}$). Hence, the dual automorph of a_{N-p}/a_{N-q} is $a_{(N-q-1)-(N-p)}=a_{p-(q-1)}$. Therefore follows the result:

if $p>q$ or $a_p>a_q$ then, $(a_p/a_q)'$ is the predecessor of a'_p/a_q' ... (i)

Case 2. Let $p<q$ or $a_p<a_q$.

Then, a_p/a_q is the dual automorph of a_p relative to the lower segment of a_q which is $a_{(q-1)-p}$. And, a'_p/a'_q which is the same as a_{N-p}/a_{N-q} is the dual automorph of a_{N-p} relative to the upper segment of $a_{N-q}=a_{N-(q-p)}$. And since $(a_p/a_q)'=a_{N-(q-1-p)}=a_{N+p-q+1}$, follows the result.

if $p<q$ or $a_p<a_q$, then $(a_p/a_q)'$ is the immediate successor of a'_p/a_q'. ... (ii)

A combination of (i) and (ii) is Theorem 1.

The relation between b/a and a/b is the content of the next theorem.

THEOREM 2. *b/a is the immediate successor or the immediate predecessor of $(a/b)'$ according as $b>a$ or $b<a$.*

Proof. Proceeding as before and taking the 2 cases when $p>q$ and $p<q$, the following two results can be proved.

If $p>q$, $(a_q/a_p)'$ is the immediate successor of a_p/a_q. ... (i)

If $p<q$, $(a_q/a_p)'$ is the immediate predecessor of a_p/a_q. ... (ii)

A combination of (i) and (ii) is theorem 2. Combining theorems 1 and 2, we have.

THEOREM 3. *Whether $b>a$ or $b<a$ or $b=a$, $b/a=a'/b'$.*

Proof. for, let $b>a$. Then since by theorems 1 and 2 b/a is the immediate successor of $(a/b)'$ and $(a/b)'$ is the immediate successor of a'/b', it follows that $b/a=a'/b'$. Similarly for $b<a$.

THEOREM 4. *b'/a is a symmetric function of a and b.*

For, since by theorem 3, $b'/a=a'/b''=a'/b$, b'/a remains the same when a and b are interchanged.

§ 3. *The Group Operation in c.*

It shall next be proved that the symmetric function $b'/a=a\oplus b$ is a group operation in the finite chain lattice c.

THEOREM 5. $a\oplus b=b\oplus a$.

This is a consequence of theorem 4.

THEOREM 6. *$a\oplus b=a$ if and only if $b=a$.*

For, by theorem 4, $a+0=0+a=a'/0=a$.

Again, if $a'/0=a$ the dual automorphism of a' in the whole chain $=a$. Hence etc.

THEOREM 7. *The operation $a\oplus x$ on x as x varies, is a cyclic permutation of the elements of the chain.*

Proof. $a+o=a'/o=a$. And $a'/1=a-1$. Clearly, the element a_M/a_x as x varies from 0 to N − M has the values

$$a_M/a_O=a_{N-M},\ a_M/a_1=a_{N-(M-1)},\ \cdots\cdots$$
$$\cdots\ a_M/a_{M-1}=a_{N-1},\ a_M/a_M=a_N,\ \cdots$$
$$\cdots\ a_M/a_{N-(M-1)}=,a_M/a_{N-M}$$

Thus the operation $a_p\oplus x$ as x varies over all the elements a_1, $\cdots$, a_N effects a cyclic permutation of

$$(a_0\ a_1\ a_2,\ \cdots,\ a_p\ a_{p+1},\ \cdots,a_N)$$

to $(a_p\ a_{p-1}\ a_{p-2},\ \cdots\ a_0\ a_N,\ \cdots,\ a_{p+1})$

Hence since the operation $\oplus$ is a cyclic permutation and this is known to be associative, it follows that the operation $\oplus$ is *also associative.*

Thus from theorems 5, 6, 7 and the above result, is established.

THEOREM 8. The set of $N+1$ elements of the finite chain lattice forms a cyclic group of order $N+1$ with a_0 as identity element under the group operation $\oplus$ as defined above.

§ 4. *The Algebra of Divisors.*

The Algebra of Divisors is a chain lattice which is the direct product of chains. Thus if $N=p_1^{N_1}\ p_2^{N_2} \cdots p_r^{N_r}$ then the algebra of divisors is the direct product of the lattice containing N_1+1, $N_2+1, \cdots, N_r+1$ elements respectively and the algebra can thus be exhibited as a group which is the direct product of cyclic groups of orders N_1+1, $N_2+1, \cdots\cdots$.

If D and D′ are two divisors of N,

$$D = p_1^{d_1}\ p_2^{d_2} \cdots\cdots p_r^{d^r}$$

$$D' = p_1^{d'_1}\ p_2^{d_2'} \cdots\cdots p_r d_r'$$

and if $d_1+d_1' > Ni+1$,
taking c_i as the smallest positive residue of $d_i = d_i'$ (mod $Ni+1$),
then $D \oplus D' = p_1^{c_1}\ p_2^{c_2} \cdots\cdots p_r^{c_r}$.

The identity element here is the divisor and it is easily seen that $\oplus$ is commutative and associative.

GLEANING

Electrical Manufacture

Mathematical activity is most intense in

(1) studying structural and dynamic problems such as strain, creep and fatigue.

(2) apprising the evil effects of sudden loads etc., and studying methods of minimising them.

(3) studying system performance particularly the most effective or economic location of proposed new equipment.

(4) refining the design of generators, motors, transformers etc., so as to improve their electrical efficiency and reliability and in similar improvement of the thermal efficiency of turbines.

From, *The Annamalainagar Mathematics Exposition.*

SYMPOSIUM ON MODERN ALGEBRA AND THEORY OF ELEMENTARY PARTICLES

Chairman: DR. B. S. MADHAVA RAO

Opening speech by the Chairman.

CONTENTS

1. Introduction
2. Algebra and quantum mechanics
3. Concept of elementary particles
4. Representations of the Lorentz group and the general theory
5. Hypercomplex numbers and the particle aspect
6. Guage group and particles of zero rest mass
7. Charge invariance and neutral particles
8. Subtraction formalism

§ 1. Introduction

The ever increasing use of the deepest, and the most abstract parts of mathematics has been a remarkable feature of modern physical theory. Mathematics of the present day abounds with abstract notions like abstract sets, abstract spaces, abstract algebra, abstract analysis and so forth, and equally so in the highest departments of modern physics like relativity and quantum theory it appears that the abstract point of view is likely to yield the most fruitful results. It is also true that the more varied and more subtle contributions from mathematics are in proportion to the wider range of physical fact acquired as a result of more and more accurate experimental research.

This increase in the application of abstract mathematical notions can be traced to a great extent to the change brought about by quantum mechanics in the meaning to be attached to physical quantities. In the old classical physics a physical quantity was considered the exact equivalent of the mathematical function assigned to the observable, or that it was the number which the function took on at specified times. Mathematics now abounds with concepts which are not functions and yet allow the assignment of numbers under certain conditions for e.g., matrices, differential operators, integral

operators, groups, and tensors. In view of this, quantum mechanics asks us to take these possibilities more seriously than classical physics did, and says that the definition of physical quantities as operators should be taken, not in a symbolic sense, but in an exact sense. With this interpretation an operator may yield not one but a large set of numbers which can however be consistently used with the aid of a statistical theory. The change brought about by quantum mechanics consists therefore in the generalisation that to one quantity there may correspond many elements of nature. In the language of mathematics, we might say that the number concept has been replaced by the more general concept of aggregates.

§ 2. Algebra and quantum mechanics

This widening of the concepts of theoretical physics is bound to be of significance to the mathematician also. Among mathematicians there are some to whom the idea of the application of their science to other fields is repugnant; there are many who are indifferent as to whether their results are applied or not, and again a few who appear to value the importance of their results in proportion to the number of applications that can be made of them. To whatever category one may belong, every mathematician will probably agree that the applications have a way of stimulating questions which spur the progress of mathematics. Indeed the history of the development of mathematics is a clear witness to this. In some branches like analysis one can even say that mathematicians owe more of their advance in those branches to the physicist than to any other agent. We thus recall how the problem of wave motion and heat conduction led to the development of the function concept, and the introduction of orthogonal series which are basic elements of present day analysis. Dirichlet's problem in potential theory had a profound influence on the calculus of variations and led to Fredholm's Theory of integral equations. In the masterly hands of Hilbert it became a theory of orthogonal transformations and reduction of quadratic forms, and created the atmosphere which stimulated the basic discoveries of F. Riesz on function spaces including the abstract Hilbert space. It is a remarkable but a common feature in physical theory that while these developments, excited by physics, were taking place in mathematics the physicist himself had little or no interest in them until the advent of quantum mechanics compelled him to look at them more seriously. It is common knowledge nowadays that the analytical problems of quantum mechanics can be thought of in terms of linear transformations in an abstract Hilbert space.

On the other hand, the relationship between algebra and physics has been rather a loose one. It is no doubt true that the theory of groups has always played a role in theoretical physics, but this role has been often a subordinate or implicit one. Thus analysis of space and time has involved group-theoretic considerations, and classical dynamics has employed group methods through transformation theory, and the recent studies of Poincaré and Birkhoff have freely used topological notions. But it is only in recent years that the more profound portions of the theory of groups and some other parts of modern algebra have played notably enhanced roles in relativity and quantum theory. In this case it is perhaps not true, as in the case of analysis, that these branches of group theory and algebra were inspired by the physicist. The researches of Frobenius, Schur, and Weyl on the theory of group representations, of Noether, Weathurburn, Dickson, Artin and others on abstract algebras, and of Cartan on continuous groups were carried on independantly as a result of the fusion of several fundamental ideas in mathematics itself. These have now found application in quantum mechanics and this relationship is not only attractive from many points of view, but it is also fair to admit that in recent years these very branches of mathematics have made advances in some directions as a consequence of this relationship to quantum mechanics. The following are a few typical examples :—

(1) Semi-simple Lie groups—Rotation and Lorentz groups are particular cases of these, and the properties of these special groups and their representations have suggested advances in the general case also.

(2) The Jordan quantum-mechanical algebra—

Jordan suggested that the statistical properties of quantum mechanics are expressed most simply in terms of a hypercomplex algebra which is commutative but not associative unlike the ordinary non-commutative algebra of matrices of the q-numbers. The study of the structure of such Jordan or r-number algebras shows that, with one single exception, all the irreducible r-number algebras are equivalent to algebras obtained by quasi-multiplication of real matrices. Quasi-multiplication is defined by $ab = \frac{1}{2}(a{\cdot}b + b{\cdot}a)$ where $a{\cdot}b$ is the ordinary matrix product. Let A′ be an associative but not necessarily commutative algebra having real numbers as coefficients. By Wedderburn's theorem we can select a matrix algebra for A′. A sub-system A of A′ closed for $a \pm b$, λa, and $ab = \frac{1}{2}(a{\cdot}b + b{\cdot}a)$ but not necessarily

for $a{\cdot}b$ might exist, and can be shown to be an r-number algebra. Such an algebra is said to result from quasi-multiplication, and is in essence a matrix algebra. The r-number algebras of Jordan with a single exception are all of this type. This exception is the algebra of all three-rowed Hermitian matrices with elements in the real non-associative algebra of Cayley numbers.

It has further been shown by Albert that this exceptional algebra is not equivalent to any algebra obtained by quasi-multiplication of real matrices.

Later Albert has generalised special types of non-associative algebras like the Jordan algebra, and obtained a general theory of non-associative algebras.

(3) Rings of operators.

The attempt to elucidate some aspects of quantum mechanics leads to the point of view that operators in an abstract Hilbert space might be considered as constituting a ring. This gives an approach to a class of problems dealing with abstract algebras without a finite basis, different from all types hitherto considered.

(4) Logic of quantum mechanics.

New methods of abstract lattice theory result from the discussion of the logical structure of quantum and classical mechanics. It has been shown that the propositional calculus of the former has the same structure as that of an abstract projective geometry, while the physically significant statements concerning a given dynamical system in classical mechanics constitute a Boolean algebra. In classical mechanics any propositional calculus involving more than two propositions can be decomposed in the sense of modern algebra, while quantum theory involves irreducible propositional calculi of unbounded complexity. Thus quantum mechanics has a greater logical coherence.

(5) Unitary representation of the inhomogeneous Lorentz group.

Operators denoting displacements and rotations in spacetime are linear because the equations are linear. These operators form a representation of the group of possible co-ordinate transformations which is the inhomogeneous Lorentz group. There is a unique correspondence between the possible Lorentz-invariant equations of quantum mechanics, and the representations of this group of operators. This extension of the Lorentz group to the inhomogeneous

case is suggested by the work of Majorana on the possibility of setting up a theory for an elementary particle of spin ½, and of Dirac on photons.

(6) Ray representations in quantum mechanics.

Since the wave functions ψ are known and have a physical meaning only to within a constant factor of absolute value 1, the discussion of group representations should be based on those to within a factor unity. These are called ray-representations. If the group of symmetry operations is a continuous (Lie) group, the representations may be discussed by considering the infinitesimal representations of the group. The commutation properties characterising these will then be valid to within an additive constant multiple of unity. There exist some groups for which these extra terms are not trivial and seem to play an essential role.

It appears that these ray-representations are closely related to the notion of the gauge-group so important in the theory of elementary particles of zero rest mass.

(7) Simply reducible groups and their representations.

The symmetric permutation groups of the third and fourth degrees, the quaternion group, the three dimensional rotation group, and the two dimensional unimodular unitary group are among the groups which are of importance for quantum mechanics. The study of the properties common to the representations of these groups gives rise to a more inclusive group called the simply reducible group defined such that

(*a*) every element is equivalent to its reciprocal,

(*b*) the Kronecker product of any two irreducible representations of the group contains no representation more than once.

Also the groups of most eigen value problems occurring in quantum theory can be shown to be simply reducible.

These examples serve to show over what a wide range the points of contact between modern algebra and quantum mechanics are distributed. Some other domains not mentioned above are molecular physics, crystal physics, and chemical physics where also group theory plays important parts. It was originally intended to take "modern algebra and quantum mechanics" as the subject for this symposium, but it was felt that it would be more useful if only a

limited domain were chosen. I have therefore chosen the subject of elementary particles of nature not only because it holds the centre of interest in modern physics, but it is also the one where modern algebra seems likely to play an important role.

§ 3. Concept of elementary particles.

The oldest known elementary particles are the *electron* and the *proton.* The facts about these established by experiment are: the electron is associated with the minimum unit of negative electricity $-e$, always associated with the same mass m, and has a spin and associated with it the same constant angular momentum $h/2$. The proton has a positive charge e, is the smallest known unit of positive electricity, is always associated with the same mass $M \sim 2000\ m$, and has a spin angular momentum $h/2$.

The simple picture of electrons and protons being constituents of all material structures including nucleii was rendered untenable as a result of further progress in experiment and theory. The presence of an electron in the nucleus is inconsistent with Dirac's relativistic theory of the electron. Experimental methods which had been devised to determine not only the spin but also the kind of statistics the nucleii satisfy led to the two rules:

(1) nucleii of even atomic weight have integral spin, and nucleii of odd atomic weight have half-integral spin.

(2) the former nucleii satisfy the Einstein-Bose statistics, and the latter the Fermi-Dirac statistics.

These results appeared inexplicable on the basis of a proton-electron constitution of nucleii according to which the spin and statistics appeared to depend only on the number of protons in the nucleus, the electrons contributing nothing to either.

The discovery of the *neutron* having nearly the same mass as the proton, no charge, a spin $h/2$, and satisfying the Fermi-Dirac statistics provided a solution of these difficulties. The assumption that nucleii are built up of proton and neutrons is consistent with the spin, statistics, mass and charge of nucleii. The phenomenon of β-decay which appeared quite natural on the old picture of proton-electron constitution of nucleii had however to be fitted into the new scheme. This could only be done by postulating that whenever the phenomenon takes place, a neutron turns into a proton giving birth to an electron much as a quantum of light is emitted

by an atom. Conservation of angular momentum or spin requires that in the process there should also be emitted another particle of spin $h/2$ and no charge. This particle whose mass can be shown to be small compared with that of the electron, if not exactly zero, was first postulated by Pauli and is called a *neutrino*. Its experimental determination has not yet been conclusively achieved. The process of β-decay is then described by

$$\text{neutron} \rightarrow \text{proton} + \text{electron} + \text{neutrino}.$$

Thus while the neutron cannot be thought of as a composite particle made up of a proton and electron since it would then have an integral spin, and satisfy the Einstein-Bose statistics, the above process requires a radical revision of the concept of elementary particles in that it makes it impossible to think of the neutron as permanent and immutable.

The next elementary particle to be discovered, the *positron* with the same mass, spin, statistics as the electron but a charge $+e$ also supported this revised concept. On the theoretical side, the positron had been postulated by Dirac on the basis of his relativistic wave equation of the electron. The solution of the difficulty of negative energy states which appear in relativistic mechanics was provided by the well-known hole theory. This theory which is successful because the electrons satisfy the Fermi-Dirac and not the Einstein-Bose statistics, also provided natural and simple explanations of the phenomena of pair creation and annihilation. The discovery of the positron by Anderson, and the demonstration of the pair creation process in cosmic ray showers by Blackett have provided the most remarkable confirmation of these two predictions of the Dirac theory, and supported the idea that the elementary particles cannot be considered as permanent and immutable.

Further experimental facts on nuclear forces, *e.g.*, forces acting between a proton and a neutron showed that these forces must be of short range *i.e.* they must diminish with distance far more rapidly than the electrostatic force between two charged particles and in addition be about a hundred times stronger. Using the general principles of relativity and quantum mechanics it can be shown that the intermediate field through which such forces act are connected with particles whose rest mass is about 170 times the electron mass. The postulation of such a particle of charge $\pm e$ originally occurred in a theory put forward by Yukawa. To explain the exchange nature of nuclear forces *i.e* a force which changes a

proton into a neutron, and a neutron into a proton every time the two particles interact, Yukawa postulated that the interaction takes place through the intermediary of these particles of charge $\pm e$ and mass μ called *mesons*; thus

$$\text{proton} \rightleftarrows \text{neutron} + \text{positive meson}$$

$$\text{neutron} \rightleftarrows \text{proton} + \text{negative meson.}$$

The discovery of particles in cosmic radiation with a mass of this order and both signs of charge must be regarded as a confirmation of the general idea of the Yukawa theory. The β-decay is fitted into this theory by assuming that the original processes take place really in two steps. The first step is the one above, the mesons being constantly emitted and reabsorbed by the heavy particles for periods so short that they could never be observed, and the second step is

$$\text{positive meson} \rightarrow \text{positron} + \text{neutrino}$$

$$\text{negative meson} \rightarrow \text{electron} + \text{neutrino,}$$

and corresponds to a spontaneous decay of the meson. This instability of the meson introduces a new feature into the concept of an elementary particle. The spontaneous decay of the meson to an electron and a neutrino in a period of 10^{-6} sec. can be proved from an analysis of experiments on cosmic radiation. It might also be mentioned that there is some evidence, though not conclusive, of the existence of a *neutral meson* in cosmic radiation.

Relativistic quantum mechanics predicts that an *anti-proton* (i.e. mass M but charge $-e$) should exist in nature, and be capable of being created in pairs with a proton. It can however be shown that the probability of the creation of such pairs by the materialisation of a quantum of radiation is at least a million times smaller than the corresponding process for an electron. A negative proton has not yet been detected. Another type of elementary particle postulated by Bhabha viz. a *heavy particle of charge 2e or $-e$* but mass higher than that of the proton has also not yet been detected.

To sum up, we might say that all these particles are elementary in the sense that all structures in nature are built out of them while they themselves cannot be considered as built up of more fundamental units. However they are not immutable for all time, but take part in a considerable number of transformations among themselves in which the number of particles of any given type may change.

The only quantities conserved in all these reactions are energy, momentum, angular momentum, and charge.

It is also necessary to mention that the ordinary electromagnetic, and the gravitational field (in the case of weak fields) can be dealt with on the basis of the principles of quantum mechanics, and can be connected with particles viz. the *photon*, and the '*graviton*' with their rest masses zero, and integral spins.

We can characterise elementary particles by three important parameters viz. rest mass, charge, and spin (leaving aside the magnetic moment), including the singular cases where these might have zero values.

§ 4. Representations of the Lorentz group, and general theory.

In recent years it has been found possible to set up a general theory which can be used consistently so as to include all the types of elementary particles mentioned above. This theory which satisfies the demands of both relativistic invariance, and the quantum hypothesis is called the field theory of relativistic quantum mechanics. It consists essentially of two stages in its application. The first part where a wave field is set up to represent the particles, the wave functions and the field equations satisfying the postulates of relativity, is called the c-number theory. To obtain the particle picture, and to make a transition to the many particle problem which is necessary because of the fact that the elementary particles are not permanent and immutable, the next part is the quantum-mechanical one where the field functions at different points are not treated as commuting with each other unconditionally. The expression of non-commutation gives rise to the quantum conditions, and this method of quantisation which explains the particle characteristics is called the q-number theory. Such a passage from the c-number to the q-number theory, a transition from a one particle to a many particle picture, is called double quantisation.

The essential steps in the c-number theory are

(1) the setting up of a Lagrangian function L which is to be invariant for the transformations of the (proper) Lorentz group,

(2) the derivation of the field equations

$$\sum_k \frac{\partial}{\partial x_k} \left\{ \frac{\partial L}{\partial q_k^{(r)}} \right\} - \frac{\partial L}{\partial q^{(r)}} = 0 \ ; \ q_k^{(r)} = \partial q^{(r)} / \partial x_k ; \ q^{(r)} = q^{(r)}(x).$$

from the variation principle

$$\delta \int L\,(q_k^{(r)},\ q^{(r)})\ d^4x = 0,$$

(3) the setting up of the energy-momentum tensor formed from L by

$$T_{ik} = \sum_r \left(\frac{\partial L}{\partial q_k^{(r)}}\right) q_i^{(r)} - L\delta_{ik},$$

just as the Hamilton function is formed from the Lagrangian in classical mechanics. T_{ik} satisfies the continuity equation and the components $-T_{44}$ and $-iT_{k4}$ $(k = 1, 2, 3)$ give the energy density, and momentum density respectively. T_{44} need not in general be positive definite.

(4) the setting up of the angular momentum tensor satisfying the continuity equation.

(5) when the field quantities are divided into the complex quantities $U(x)$, their conjugates $U^*(x)$, and the real quantities $V(x)$ one has to postulate the invariance of L against gauge transformations, and this leads on to the possibility of setting up of the current vector.

$$s_k = \varepsilon i \left\{ \frac{\partial L}{\partial U_k^{(r)}} U^{(r)} - U^{*(r)} \frac{\partial L}{\partial U_k^{*(r)}} \right\}, \quad (\varepsilon = \text{a constant})$$

which also satisfies the continuity equation. s_i $(i = 1, 2, 3)$ gives the current and is_4 the charge density.

Suitable modifications can be consistently made to take account of the presence of external electromagnetic fields which, along with the gravitational field, occupy special positions in that both are classical fields, and the rest mass associated with each of them is zero.

Of the three fundamental properties the mass, charge, and spin of elementary particles, it is remarkable that the spin, which is perhaps not the simplest one from the experimental point of view, comes up first in the mathematical formalism. In the frame work of the c-number theory sketched above we have made no mention of the field quantities themselves, but the knowledge of the nature of these quantities is necessary to derive further conclusions about energy density and charge density. Since the field equations are derived from a Lagrangian, invariant under transformations of the proper Lorentz group, it follows easily that the field quantities should transform according to irreducible representations of this group defined as the group of transformations keeping the form $\sum x_k^2 = \vec{x}^2 - x_0^2$ invariant, having the determinant $+1$, and not reversing the direction of time. Such quantities are, as is well known, the spinors, and the field quantity can be written as $U(j,k)$

characterised by two indices j and k corresponding in spinor notation to spinors with $2j$ undotted, and $2k$ dotted indices, and symmetric in them. The most important operation relating to these quantities is the reduction of the product $U_1(j_1,k_1).U_2(j_2k_2)$ which according to the well-known Clebsch-Gordan rule decomposes into several $U(j,k)$ where independently of each other j and k run through the values

$$j=j_1+j_2,\ j_1+j_2-1,\ \cdots\ |j_1-j_2|$$

$$k=k_1+k_2,\ k_1+k_2-1,\ \cdots\ |k_1-k_2.|$$

For the sub-group of space rotations the distinction between dotted and undotted indices disappears, and $U(j,k)$ behave like the product $U(j).\ U(k)$. This product decomposes into several $U(l)$ each having $(2l+1)$ components with

$$l=j+k,\ j+k-1,\ \cdots\ |j-k|.$$

Under space rotations $U(l)$ with l integral gives one-valued representations, and for l half-integral double-valued representations. This along with the fact that for a particle of spin f there are $(2f+1)$ independent states (for a given energy, and momentum), which transform among themselves under space rotations, suggests that we should call $(j+k)$ the spin of the particle. This would however involve not one spin value but a number of others. To get a satisfactory theory one should set up a relation between spin value and the number of independent plane waves possible for a given wave vector. This can be done by setting up suitable additional conditions to be satisfied by the quantities, so chosen that in the rest system there exists only one irreducible group of $(2f+1)$ components going into each other under space rotations. It then follows from Lorentz invariance that for any arbitrary system of reference $(2f+1)$ eigen functions always belong to a given k_i (wave vector).

Thus for particles of integral spin $(2j+2k=\text{even number})$ we can set up such a theory using the ordinary world tensors, and for half-integral spin $(2j+2k=\text{odd number})$ spinors which cannot be reduced to such. Working with these general field quantities, and setting up the energy-momentum tensor and the current vector we find some very general results:

(1) For particles of half-integral spin, the total energy is not necessarily positive.

(2) For particles of integral spin the charge density is not necessarily positive.

The results are of a purely negative character, but a deeper analysis yields the following positive results:

(1) In the case of integral spin the total energy can be made positive using suitable subsidiary conditions. As for the energy density, it is unique and positive definite only for the cases $f=0$ and $f=1$. For $f>1$ it is neither uniquely determined nor positive definite.

(2) In the case of half-integral spin the total charge is unique and positive. As regards charge density, it is unique only for the case of spin $f=\frac{1}{2}$, for which again it is the total charge that is positive definite, the result of charge density being positive definite being true only in the rest system, but not extensible to any system since it is only the charge and not the charge density that is relativistic invariant.

It therefore appears that the uniqueness and positive definite character of charge and energy density is characteristic for the lower spins 0,½, and 1. But this does not necessarily exclude the higher spins since a consistent theory might be possible for them taking interaction with external fields into consideration.

This is as far as one can go with the *c*-number theory only. The fact that the total energy of the field can be negative for particles of half-integral spin has consequences which would bring the theory in conflict with nature. Again the possibility of charge density being negative for particles of integral spin makes the idea of particle or probability density untenable, and a transition to a many particle picture becomes necessary. Also such a transition should explain the fact that while total charge in a given volume is measurable, it cannot take on any value positive or negative but only values $\pm\, ne$. Not only the charge but also the energy has to be quantised i.e. its eigen values are to be integral multiples of $h\nu_s$. All these suggest the need for the quantisation of the field. From general considerations also, one can see the need for this. In fact there is no special reason why the field quantities should commute with each other, there being even in the *c*-number theory examples of some quantities which do not commute (*e.g.* Dirac matrices, infinitesimal rotation operators etc.) Also there must be some conditions which the statements about non-commutation should satisfy. These rules should for example be consistent with the field equations since they (the commutation rules) are only statements restricting the field quantities in that they do not allow the consideration of all arbitrary field functions satisfying the field equations. Hence the

expressions of the rules must be invariants of the motion, and from the nature of the restrictions imposed they should be relations between canonical coordinates and momenta. Being invariants of the motion they are expressible in terms of the bracket expressions

$$\left[a, b\right]^{\pm}$$

where, as is well known, the $\pm$ is to be taken according as the particles satisfy the Fermi-Dirac or Einstein-Bose statistics, and are of the form

$$\left[U(x), U^*(x')\right]^{\pm} = D(x, x'),$$

the transformation properties of the U's under Lorentz transformations also requiring that the D's should transform in a certain way. We further require of the function D that (i) it should be a function only of the invariant distance between the points, (ii) that $D=0$ if the points are separated by a space-like distance i.e. if each lies outside the light cone of the other, since signals cannot be propagated with a velocity greater than that of light. These above conditions are together stringent enough to determine the form of the function D almost uniquely.

By merely considering the algebraic nature of the transformation of the D's under Lorentz transformations in the cases of integral and half-integral spin one is led to the following general results :—

(1) For integral spin quantisation according the Fermi-Dirac statistics is impossible.

(2) For half-integral spin there is no algebraic contradiction in either statistics being satisfied, but the removal of the negative energy difficulty is not possible if one uses the Einstein-Bose statistics.

This well-known connection between spin and statistics not properly explained till now is thus seen to be one of the most important applications of the theory of representations of the Lorentz group.

The above results are again of a negative character but the actual carrying through of the quantisation shows that:

(1) for half-integral spin a satisfactory theory can be obtained by quantisation with the Fermi-Dirac statistics, and leads to the consequence that the total charge for a state $\vec{k}$ (wave vector) is 0 or e.

In the former case the momentum and energy of the state are zero, and for the latter case the total momentum is $\vec{hk}$, and the energy is $\pm hk_0$ for the positive and negative states. Also hX (the wave equation being $\Box u = Xu$) can be interpreted as the rest mass of the particle.

(2) for integral spin a satisfactory theory can be obtained by quantisation with the Einstein-Bose statistics, and leads to the result that the total charge may be $\pm ne$ (n being a positive or negative integer or zero) while for both cases the corresponding total energy is hk_0. Also the rest mass can be shown to be hX.

§ 5. Hypercomplex numbers and the particle aspect.

The general theory of particles of integral and half-integral spin sketched above is based on field quantities which are spinors satisfying a wave equation $\Box\, a^{\dot{\lambda}\dot{\mu}\ldots}_{\delta\rho\ldots} = X^2\, a^{\dot{\lambda}\dot{\mu}\ldots}_{\delta\rho\ldots}$, and in order that the theory may represent particles of a single spin it is essential that further symmetry restrictions be imposed on the spinors. This is achieved by assuming the spinor to be symmetric in all the undotted and dotted indices, and further that it satisfies a divergence condition equivalent to assuming that the spinors $p^{\dot{\nu}\rho}\, a^{\dot{\lambda}\dot{\mu}\ldots}_{\rho\delta\ldots}$ and $p_{\dot{\nu}\delta}\, a^{\dot{\nu}\dot{\mu}\ldots}_{\delta\rho\ldots}$ are also symmetrical. This makes it possible to pass from the second order wave equation to a system of first order wave equations of the type

$$\left.\begin{aligned} p^{\dot{\nu}\rho}\, a^{\dot{\lambda}\dot{\mu}\ldots}_{\rho\delta\ldots} &= X\, b^{\dot{\nu}\dot{\lambda}\dot{\mu}\ldots}_{\delta\ldots\ldots} \\ p_{\dot{\nu}\rho}\, b^{\dot{\nu}\dot{\lambda}\mu\ldots}_{\dot{\delta}\ldots}, &= X\, a^{\dot{\lambda}\dot{\mu}\ldots}_{\rho\delta\ldots} \end{aligned}\right\} \quad (p^{\dot{\nu}\rho} = \text{gradient spinor}).$$

where the spinor b satisfies the same conditions as the spinor a. The cases where a has $2k$ undotted and $(2k-1)$ dotted indices or $2k$, dotted and $2k$ undotted indices according as the spin is half-integral or integral respectively are specially simple, and denoting the spinors in this case by $a^{(0)}$ and $b,^{(0)}$ they have the property of going over into each other by reflections. In the integral case $a^{(0)} = b^{(0)}$ and in the half-integral case $a^{(0)} \rightleftarrows b^{(0)}$ under reflections so that together they form a system invariant for the full Lorentz group, and the first order wave equations in them may be said to be in a form leading to the well known Dirac type of wave equation. The wave functions (viz. the spinors) involve a number of dotted and undotted indices, but it is easy to show in general that they can be reduced to a type

in which only one spinor index is involved. The spinors so far considered are all symmetric and hence are given when one knows the number of undotted and dotted indices which are equal to 1, and we can set up a correspondence between a spinor $a^{\dot{\lambda}\dot{\mu}}_{\rho\delta}$ with $2k$ undotted, and $2l$ dotted indices, and a magnitude $A^{\dot{r}}_{s}$ whose indices $\dot{r}$, s indicate how many of the dotted and undotted indices are equal to $\dot{1}$, 1 respectively. To obtain this correspondence one has to find out a matrix which transforms the spinor indices to the indices $\dot{r}$, s. If the spinor be symmetric the correspondence is (1, 1) but not so for a non-symmetric spinor. Since a spinor without symmetry properties transforms according to a product representation of the Lorentz group which contains the factor $\mathfrak{D}_{\frac{1}{2},0}$ $2k$ times, and the factor $\mathfrak{D}_{0,\frac{1}{2}}$ $2l$ times while $A^{\dot{r}}_{s}$ is irreducible, it follows that the correspondence of such a quantity with a non-symmetric spinor means the picking out of an irreducible sub-space and one with the maximum dimension number from the representation corresponding to the spinor. This reduction can be shown to depend essentially on the reduction of the product representation $\mathfrak{D}_{\frac{1}{2}} \times \mathfrak{D}_{k}$ of the rotation group. $\mathfrak{D}_{\frac{1}{2}}$ and $\mathfrak{D}_{k}$ are characterised by their infinitesimal transformations σ^l and α^l (k), and the product representation can be characterised by the matrix

$$A\ (k) = \sum_{1}^{3} \sigma^l \times \alpha^l\ (k).$$

This matrix commutes with all the matrices of the product representation as can be shown easily by writing out the Pauli matrices σ^l and the matrices α^l (k). It is a well-known result of the general semi-simple Lie groups that if such a matrix commuting with all the matrices of the infinitesimal product representation is brought to the diagonal form, the product representation is simultaneously split up into its irreducible components. The reduction of $\mathfrak{D}_{\frac{1}{2}} \times \mathfrak{D}_{k}$ is therefore related to the principal axis transformation of A (k). Let U be the matrix such that UAU^{-1} is the diagonal matrix and be written in the form

$$U = (2k+1)^{-\frac{1}{2}} \begin{pmatrix} u_1\ (k+\frac{1}{2}) & u_2\ (k+\frac{1}{2}) \\ v_1\ (k) & v_2\ (k) \end{pmatrix}$$

$$U^{-1} = (2k+1)^{-\frac{1}{2}} \begin{pmatrix} v^1\ (k+\frac{1}{2}) & u^1\ (k) \\ v^2\ (k+\frac{1}{2}) & u^2\ (k) \end{pmatrix}$$

where u_1 (k), u_2 (k) are rectangular matrices with $(2k+1)$ rows and $(2k)$ columns; and v_1 (k) v_2 (k) are similar ones with $2k$ rows and $(2k+1)$

columns. Using the fact that the matrix U brings A (k) to the diagonal form, we get among other relations the important property of the u, v matrices viz.

$$u^{\mu}(k)u_{\mu}(k-\tfrac{1}{2})=v^{\mu}(k-\tfrac{1}{2})v_{\mu}(k)=0.$$

This shows that the rectangular matrices $u^{\mu}(k)\, u^{\nu}(k-\tfrac{1}{2})$ and $v^{\mu}(k-\tfrac{1}{2})\, v^{\nu}(k)$ are spinors symmetric in μ and ν. Forming the operators

$$R^{\mu_1, \mu_2 \ldots \mu_{2k}}(k)=v^{\mu_1}(\tfrac{1}{2})\, v^{\mu_2}(1)\ldots v^{\mu_{2k}}(k)$$

$$P^{\nu_1, \nu_2 \ldots \nu_{2k}}(k)=u^{\nu_1}(k)\, u^{\nu_2}(k-\tfrac{1}{2})\ldots u^{\nu_{2k}}(\tfrac{1}{2}),$$

we conclude from the above property that these operators are symmetric in all the indices $\mu_1, \mu_2, \ldots \mu_{2k}$. With the help of the operator $P_s(k)^{\gamma\mu\ldots\nu}$ where s is the index specifying the number of spinor indices equal to one, we can make correspond to each symmetric spinor $a_{\gamma\mu\ldots\nu}$ of rank $2k$ a magnitude A_s (s running through $2k+1$ values) and conversely with the aid of $R_s(k)_{\delta\rho\ldots\lambda}$ we can make correspond a symmetric spinor to every A_s according to

$$\left.\begin{aligned} P_s^{\gamma\mu\ldots\nu}(k)\, a_{\gamma\mu\ldots\nu} &= \sqrt{(2k)!}\, A_s \\ R_{s,\gamma\mu\ldots\nu}(k)\, A_s &= \sqrt{(2k)!}\, a_{\gamma\mu\ldots\nu} \end{aligned}\right\}$$

If the operator $P_s^{\gamma\mu\ldots\nu}$ be applied to a non-symmetric spinor the operation is not uniquely reversible; we obtain further a symmetrised spinor. Thus $P_s^{\gamma\mu\ldots\nu}$ is to be employed as a symmetrising operator, and can be generalised so as to symmetrise only a part of the indices of the spinor. Thus

$$\left.\begin{aligned} P_s^{\gamma\mu\ldots}\, a_{\gamma\mu\ldots\delta\ldots\lambda} &= \sqrt{2k\,(2k-1)\cdots}\; A_{s,\delta\ldots\lambda} \\ R_{s,\gamma\mu\ldots}\, A_{s,\delta\ldots\lambda} &= \sqrt{2k\,(2k-1)\cdots}\; a_{\gamma\mu\ldots\delta\ldots\lambda} \end{aligned}\right\}$$

Similar results hold when the index s is replaced by an index $\dot{r}$ and k by l. Having replaced a part of the indices by an index s we can replace the rest by $\dot{r}$ and obtain a magnitude $A_s^{\dot{r}}$. In particular we can form magnitudes containing only one spinor index. In this manner starting with the original spinor equations we come to the equations specified by Dirac

$$\left.\begin{aligned} p^{\tau\rho}\ \dot{\psi}^{\mathrm{A}}_{\rho} &= \chi\dot{\psi}^{\tau\mathrm{B}} \\ p^{\cdot}_{\tau\rho}\ \dot{\psi}^{\tau\mathrm{B}} &= \chi\psi^{\mathrm{A}}_{\rho} \end{aligned}\right\}$$

where A and B indicate that ψ involves magnitudes like $\dot{A}^{r}_{s}$, $\dot{B}^{t}_{p}$, and the equations are to be treated as matrix equations.

Finally the two equations can be combined into one single equation of the Dirac form

$$\partial_k\,\beta_k\,\psi + \chi\psi = 0, \quad (\partial_k = \partial/\partial x_k)$$

where β_k are matrices whose form is determined by writing the above two equations in a combined form. This is the famous form which Dirac originally derived for the wave equation of the electron; and which was really the basis of relativistic quantum mechanics; and the wave equation in this form is said to represent the particle aspect of the theory of elementary particles. The β_k are the well known Dirac matrices satisfying the commutation relations

$$\tfrac{1}{2}\,(\beta_k\,\beta_l + \beta_l\,\beta_k) = \delta_{kl}$$

and in virtue of these leads to the system of hypercomplex numbers 1, β_k, $\beta_k\,\beta_l$, $\beta_k\,\beta_l\,\beta_m$ and $\beta_1\,\beta_2\,\beta_3\,\beta_4$. The theorems relating to representations of finite groups can be extended, as is well-known, to group rings and hence also to hypercomplex numbers satisfying certain conditions. We recall some theorems relating to representations of hypercomplex systems.

(1) All representations of a completely reducible hypercomplex system with a unit element (or without a radical) are themselves completely reducible, and conversely if all representations be reducible so is the system itself.

(2) All irreducible representations of a hypercomplex system are contained in the regular representation (i. e. where the system itself is taken for the representation modulus).

(3) An absolutely irreducible system of matrices of the nth order (a system which remains irreducible by the algebraic extension of the field) which contains along with two matrices also their product, contains exactly n^2 linearly independent matrices. (Burnside's theorem).

(4) This theorem can be extended to completely reducible systems and states that the number of linearly independent matrices is equal to $n_1^2 + n_2^2 + \cdots + n_s^2$ where the irreducible parts counted only once have orders n_1, $n_2 \cdots n_s$.

(5) In the reduction of the regular representation, the representation D_ν is contained n_ν times so that $h = \sum_{1}^{s} n_\nu^2$, h being the rank of the algebra.

(6) The number of inequivalent irreducible representations is equal to the rank of the centrum.

The above theorems apply to a completely reducible hypercomplex system i. e. one which is a sum of *irreducible left ideals* and possesses a unit element. Such an algebra is also called semi-simple (an algebra is semi-simple if its radical is the zero ideal, and simple if the only proper ideal is the zero ideal).

In the case of the Dirac algebra it is extremely simple to deduce that there is only one irreducible representation (of order 4) and that every reducible representation completely splits up into irreducible representations equivalent to the one originally given by Dirac which is thus the unique one but for equivalence.

The situation however is *not so simple in the case of* the wave equation of the Dirac type for particles of spins 0 and 1. In the former case the wave equation is $\Box U - X^2 U = 0$, where U is a scalar and can be put in the form of two first order equations by writing

$$U_k = \frac{\partial U}{\partial x_k};\ \frac{\partial U_k}{\partial x_k} = X^2 U$$

or further in the Dirac *form* $\beta_k \partial u/\partial x_k + Xu = 0$ where the β_k are 5-rowed matrices, four rows operating on the vector U_k, the fifth on the scalar U. In the *latter case* which presumably corresponds to the meson, the equation $\Box U_k = X^2 U_k$ (U_k a vector) is to be supplemented by the divergence condition $\partial U_k/\partial x_k = 0$ so as to make the energy density everywhere positive definite. Alternatively, introducing a skew-symmetric tensor $U_{ik} = -U_{ki}$ with $U_{ik} = \frac{\partial U_k}{\partial x_i} - \frac{\partial U_i}{\partial x_k}$, the above equations yield $\frac{\partial U_{ik}}{\partial x_k} + X^2 U_i = 0$, and these last two might themselves be considered as the fundamental equations the first two being derived from them. The advantage is that we can put the wave equation in the Dirac form, this time however, the β_k's being 10-rowed matrices four rows of which operate on the vector U_k and six on the skew-symmetric tensor U_{kl}.

The interesting fact has been noticed that both the 5-rowed and 10-rowed matrices fulfil the commutation relationships

$$\beta_k \beta_l \beta_m + \beta_m \beta_l \beta_k = \delta_{kl} \beta_m + \delta_{lm} \beta_k.$$

The four matrices β_k, the unit matrix 1, and all powers and products *of the β_k generate 126 linearly independent elements, the number* being limited by the above commutation rules. Writing $\eta_k = 2\beta_k^2 - 1$, it can be shown that the elements 1, $M = \sum \eta_k - \sum_{k<l} \eta_k \eta_l$, $N = \eta_1 \eta_2 \eta_3 \eta_4 (1 - \sum_k \eta_k)$ commute with all the 126 elements, and that any further elements commuting with all these are linear combinations of 1, M, N. Thus the number of elements in the centrum being three, there are exactly three inequivalent representations of degrees n_1, n_2, n_3 such that $n_1^2 + n_2^2 + n_3^2 = 126$. The trivial one-rowed representation is obviously contained in this so that putting $n_3 = 1$ we have at once $n_1 = 10$, $n_2 = 5$ giving the above two representations, there being no other irreducible ones.

The proof that this algebra is semi-simple leads to some interesting considerations. Pauli has shown that this amounts to the following:—

Let e be the matrices representing the independent elements of the algebra in the regular representation and let $g_{ik} = \text{spur}\,(e_i\, e_k)$; the algebra is semi-simple if Det $\| g_{ik} \| \neq 0$.

This determinant which has 126^2 elements can be shown not to vanish. It would be interesting to find if there be an alternative method of proving this algebra to be semi-simple.

This meson algebra can be further studied on the assumption that instead of the four elements β_k there exist an arbitrary number of these elements. The representations of this generalised algebra have simple and interesting relationships with those of the original algebra, and appear to throw light on the nature of the connection between the tensor equations and the linearised wave equation of the Dirac type which reformulates them.

The question now arises as to whether a particle aspect of the theory can be set up for particles of higher spin also i.e. of writing the equations in the form

$$\partial_k \beta_k \psi + X\psi = 0,$$

with β_k as matrices having a suitable number of rows. As a preliminary to this it is necessary to derive commutation relations satisfied by the β_k's. I have recently shown that we can derive such relations by making some general assumptions:

Let the infinitesimal orthogonal transformation

$$x_k' = x_k + \sum \varepsilon_{kl}\, x_l \qquad (\varepsilon_{kl} = -\varepsilon_{lk}, \text{ numerical coefficient})$$

correspond to the transformation $\psi' = \Lambda\psi$ of the wave function with

$$\Lambda = 1 + \tfrac{1}{2} \sum \sum \varepsilon_{kl}\, s_{kl} \qquad (s_{kl} = -s_{lk}, \text{ matrices})$$

where the s_{kl} are the spin matrices. The general assumptions are

(i) the wave equation is Lorentz invariant. This leads to

$$\beta_k\, s_{lm} - s_{lm}\, \beta_k = \delta_{km}\, \beta_l - \delta_{kl}\, \beta_m.$$

(ii) $s_{kl} = k\,(\beta_k\, \beta_l - \beta_l\, \beta_k)$, ($k$ a constant)

(iii) each component s_{kl} satisfies the algebraic equation whose roots are the $(2f+1)$ eigenvalues of the spin operator.

It can be shown that these general assumptions determine the commutation rules, and the actual rules for the cases $f=3/2$, and $f=2$ have been obtained, which are however very complicated. Also the constant k appearing in the assumption (ii) remains undetermined, and although one might guess from the analogy of the electron and meson cases that $k=f^2$ ($f=$spin), the proof of this seems to depend on some deep seated property of the representations of the Lorentz group, and has not yet been given. The hypercomplex algebra corresponding to these cases of higher spins does not depend essentially on the value of k, but the further development of these algebras also appears extremely complicated.

Coming back to the case of spin one, the reduction of the representation to the three irreducible parts finds a counterpart in the reduction of the 16-component quantities $u_{\rho\rho'}$ where ρ and ρ' are indices denoting a Dirac spinor. A Dirac spinor is characterised by its being a pair of spinors transforming one into the other by a reflection through the origin, so that it is most naturally fitted when we consider the full Lorentz group. The quantities $u_{\rho\rho'}$ mentioned above are those transforming like products of such Dirac spinors, and are called *Undors*. They form a generalisation of the Dirac wave functions just as tensors form a generalisation of vectors. One could thus define a metrical undor, a gradient undor and build up a theory analogous to tensors. Just as in the case of tensors, the representations by most undors are reducible, and we can derive relations between undors of the second rank ($u_{\rho\rho'}$) and tensors. The symmetric part of $u_{\rho\rho'}$ (for which $u_{\rho\rho'} = u_{\rho'\rho}$) consists of a skew-symmetric tensor and an ordinary vector belonging to the 10-rowed representation; the anti-symmetric part splits up into a scalar with which is associated the 1-rowed representation, and a pseudo-vector and pseudoscalar for the 5-rowed representation. The question of

generalisation to undors of higher rank $u_{\rho_1\rho_2\rho_3}$, $u_{\rho_1\rho_2\rho_3\rho_4}$ etc. which can be obviously used for particles of spins 3/2, 2, etc. and with 4^3, 4^4 components etc., and the decomposition of these undors may throw light on the reducibility of the representation of the algebra associated with these higher spins.

§ 6. The gauge group, and particles of rest mass zero.

The field quantities appearing in the wave equations can be divided into the $U(x)$, their complex conjugates $U^*(x)$ considered independent of $U(x)$, and the real quantities $V(x)$. The statistical interpretation of wave functions requires that the Lagrangian L be invariant under the phase transformation

$$U^{(r)} \rightarrow U^{(r)} e^{i\alpha};\ U^{*(r)} \rightarrow U^{*(r)} e^{-i\alpha},$$

and in the case where there are no external fields, it is enough if α be assumed an arbitrary constant. In the case of an external field (electromagnetic) specified by the real potentials ϕ_k, it can be shown that a consistent theory can be obtained when there are no supplementary conditions, by replacing $\frac{\partial}{\partial x_k}$ by $D_k = \frac{\partial}{\partial x_k} - i\varepsilon\phi_k$, $D_k^* = \frac{\partial}{\partial x_k} + i\varepsilon\phi_k$, and $D_k = \partial/\partial x_k$ according as the operator is applied to $U^{(r)}$ or $U^{*(r)}$ or V respectively. The theory obtained in this manner is invariant with respect to the transformation

$$U^{(r)} \rightarrow U^{(r)} e^{i\alpha};\ U^{*(r)} \rightarrow U^{*(r)} e^{-i\alpha} \tag{1}$$

$$\phi_k \rightarrow \phi_k - \frac{i}{\varepsilon} \frac{\partial\alpha}{\partial x_k} \tag{2}$$

where α may now be an arbitrary function of position. The invariance in the force free case under constant α is sufficient to ensure that this is always correct. The prescription of invariance under gauge transformations ensures the setting up of a current vector in the case where no external fields are present or where they are present. (1) are called gauge transformations of the first kind, and (2) of the second kind.

These gauge transformations appear to play an important part when one considers the singular case of particles of rest mass zero. Taking spin one particles as the first case, since $\chi = 0$ gives photons, the equations are

$$U_{ik} = \frac{\partial U_k}{\partial x_i} - \frac{\partial U_i}{\partial x_k};\ \frac{\partial U_{ik}}{\partial x_i} = 0.$$

These together with U_{ik} remain invariant for a gauge transformation of the second kind $U_k \longrightarrow U_k + \partial f/\partial x_k$. This transformation seems to establish a fundamental difference between the cases $X=0$ and $X \neq 0$. On this basis an assumption that the photons have a very small but finite rest mass seems to be physically unsatisfactory. For, if the photon field be the external field acting, say on an electron and if $X \neq 0$ for the photon then a transformation of the kind (2) would not exist and consequently a transformation of the first kind (1) with arbitrary α would not exist for the electron which is physically meaningless. This is because (1) and (2) hold simultaneously.

We next consider the case of a particle of spin 2 with the rest mass zero described by a real field just as in the case of the electromagnetic field. The assumption of a real field means the vanishing of the current vector and charge so that the field has neither electrostatic nor electromagnetic properties. In the general theory of a particle of the single spin 2 i.e. the theory which eliminates particles of lower spins and makes the total energy positive definite, we take the field quantity to be a symmetric tensor U_{ik} satisfying the wave equation $\Box U_{ik} - X^2 U_{ik} = 0$, the spur condition $U = \sum U_{rr} = 0$, and the divergence condition $\dfrac{\partial U_{ik}}{\partial x_k} = 0$. These ensure that in the rest system there are 5 independent eigen-solutions which go into each other under space rotations so that one is actually dealing with a particle of single spin value 2. The theory can be formulated alternatively with a suitable Lagrangian L, deriving the wave equation from it by the variation principle, and building the spur and divergence of the wave equation, which method leads to the same equations as above. In the special case of $X=0$, and real fields, this Lagrangian reduces to

350326

$$L = \frac{1}{2}\frac{\partial \gamma_{ik}}{\partial x_l}\frac{\partial \gamma_{ik}}{\partial x_l} - \frac{\partial \gamma_{rk}}{\partial x_s}\frac{\partial \gamma_{sk}}{\partial x_r} + \frac{\partial \gamma_{rk}}{\partial x_r}\frac{\partial \gamma}{\partial x_k} - \frac{1}{2}\frac{\partial \gamma}{\partial x_l}\frac{\partial \gamma}{\partial x_l}; \quad \begin{bmatrix} \gamma_{ik} \equiv U_{ik} \\ \gamma = \sum \gamma_{rr} \end{bmatrix} \qquad (1)$$

which is remarkably enough the Lagrangian function of the general theory of relativity for weak gravitational fields i.e. where

$$g_{ik} = \delta_{ik} + \gamma_{ik} \qquad (2)$$

and terms of higher order than the first in γ_{ik} are neglected. The wave equation derived from the Lagrangian is

$$\Box \gamma_{ik} - \frac{\partial^2 \gamma_{rk}}{\partial x_r \partial x_i} - \frac{\partial^2 \gamma_{ri}}{\partial x_r \partial x_k} + \frac{\partial^2 \gamma}{\partial x_i \partial x_k} + \delta_{ik}\left(\frac{\partial^2 \gamma_{rs}}{\partial x_r \partial x_s} - \Box \gamma\right) = 0 \qquad (3)$$

This equation vanishes identically if the divergence be formed so that we cannot derive from it the wave equation of the second order

as in the general case. On the other hand it allows the gauge transformation of the second kind

$$\gamma'_{ik}=\gamma_{ik}+\frac{\partial\xi_i}{\partial x_k}+\frac{\partial\xi_k}{\partial x_i} \tag{4}$$

where ξ_i is an arbitrary vector field. It can also be shown that the total energy and momentum are gauge-invariant quantities. It is noteworthy that (4) arises by the change of g_{ik} under infinitesimal co-ordinate transformations

$$x_i'=x_i+\xi_i\ (x) \tag{5}$$

where ξ_i are treated as small quantities of the first order. Whereas in Maxwell's theory the gauge-invariant field quantities are obtained by once differentiating the potentials, the simplest gauge invariant quantities in the present case are the components of the curvature tensor which, up to the required approximation, can be written

$$R_{[ik]\,[lm]}=\frac{1}{2}\left(\frac{\partial^2\gamma_{il}}{\partial x_k\,\partial x_m}+\frac{\partial^2\gamma_{km}}{\partial x_i\,\partial x_l}-\frac{\partial^2\gamma_{im}}{\partial x_k\,\partial x_l}-\frac{\partial^2\gamma_{kl}}{\partial x_i\,\partial x_m}\right) \tag{6}$$

whose invariance follows immediately from (4). It is well known from general relativity theory that the general solution of (3) can be obtained as the sum of two parts one of which can be removed by a suitable gauge transformation (change of co-ordinate system) and another which satisfies the wave equation $\Box\gamma_{ik}=0$. As shown by Einstein this wave equation has exactly two components (as in the Maxwell field) for a given wave vector k_r so that they transform into each other under gauge transformations.

The γ_{ik}-field can be quantised just as in the case of the Maxwell field, and leads to gravitational quanta or "gravitons" of spin 2.

The idea of gauge transformations of the second kind can be generalised and permits the description of fields of arbitrary spin and rest mass zero. Thus in the case of arbitrary integral spin where the field quantity is a tensor $A_{ik\ldots l}$ the gauge transformation is defined by

$$A'_{ik\ldots l}=A_{ik\ldots l}+N_{ik\ldots l}$$

where

$$N_{ik\ldots lm}=\frac{\partial C_{k\ldots lm}}{\partial x_i}+\frac{\partial C_{i\ldots lm}}{\partial x_k}+\cdots+\frac{\partial C_{ik\ldots l}}{\partial x_m},$$

$$\Box C_{k\ldots l}=0,\ C_{kk\ldots l}=0,\ \partial C_{k\ldots l}/\partial x_k=0.$$

If $A_{kl\ldots}$ has f indices there exist $(2f+1)$ linearly independent waves to each wave number vector. The tensor $C_{ik\ldots l}$ has $(f-1)$ indices and there are accordingly $(2f-1)$ linearly independant plane

waves $N_{ik\ldots l}$ to each wave number vector. If we consider states which go into each other under gauge transformations as equivalent there are therefore only *two* $(\overline{2f+1}-\overline{2f-1})$ independent and really different plane waves for a specified wave number and frequency, for the case of zero rest mass. Gauge transformations can be set up with the aid of spinors in the case of half-integral spin also, and lead to the same result about two independent components. In the special case of spin 1/2 the gauge transformation is naturally absent since $2f-1=0$ and $2f+1=2$.

A more detailed investigation of these general gauge groups and a theory of representations for them would yield useful results concerning the singular case of particles of zero rest mass.

§7. Charge invariance and neutral particles.

In th previous article we have treated the limiting case of zero mass. We might now consider the other limiting case of particles of zero charge i.e. neutral particles. This corresponds to the case of real fields i.e. $U=U^*$ where the current vector vanishes identically and the associated particles cannot generate an electromagnetic field. The original form with a complex U is equivalent to two real fields $V=V^*$, $W=W^*$ with

$$U=\frac{1}{\sqrt{2}}(V+iW);\ U^*=\frac{1}{\sqrt{2}}(V-iW),$$

the numerical factor being introduced for the sake of convenience in quantisation. • A theory for neutral particles can be obtained from such a theory by striking out W, thus having only a single real V. This theory is called an "abbreviated theory".

The question arises whether such theories would be possible for all spins. Thus for spin 1 the transition to real fields is obtained by putting the field function U_k which is a vector, equal to $\frac{1}{\sqrt{2}}(V_k+iW_k)$ and identifying U_k and U_k^*. Such a theory would describe neutral mesons or *neutrettos*.

In the case of a particle of spin 1/2 this method of splitting up into two real fields requires a modification in view of the field functions being spinors. Writing the Dirac equation in the form

$$\gamma_k\left(\frac{\partial u}{\partial x_k}\right)+\chi u=0,$$

and introducing the usual α, β matrices according to

$$\alpha_k=i\gamma_4\,\gamma_k \text{ for } k=1,2,3;\ \beta=\gamma_4$$

the equation can be written in the alternative form

$$\frac{\partial u}{\partial x_k} + \vec{\alpha} \cdot \left(\frac{\partial u}{\partial \vec{x}} \right) + i \chi \beta u = 0.$$

Introducing now a matrix C defined by

$$\beta^* = -\mathrm{C}\beta\mathrm{C}^{-1} ; \ \alpha^* = \mathrm{C}\alpha\mathrm{C}^{-1}$$

it can be shown that C exists, that C*C commutes with all the γ_k and is thus a constant. Also C can be shown to be symmetric and hence so chosen that C*C = 1. This C-matrix of Pauli can be used to set up a Lorentz-invariant ordering between the solutions of Dirac's equation with positive and negative frequency viz.

$$u_-^* = \mathrm{C}u_+ ; \ u_+ = \mathrm{C}^{-1} \ u_-^*$$

and solutions of this sort are called *charge conjugate* solutions. This terminology can be justified by a consideration of the effects of an electromagnetic field, and it can be shown that if u_+ satisfies the wave equation with the charge $+e$, then u_- satisfies it with the charge $-e$.

The decomposition of the spinor field u into the "real fields" v and w is to be done according to

$$u = \frac{1}{\sqrt{2}} (v + iw) ; \ u^* = \frac{1}{\sqrt{2}} \mathrm{C} (v - iw),$$

where v and w fulfill the Lorentz-invariant reality conditions

$$v^* = \mathrm{C}v ; \ w^* = \mathrm{C}w.$$

The whole of the q-number theory can be worked out with the pair v, w in place of the original single function u. The transition to the charge conjugate state is realised by $v \to v$, $w \to -w$, and for this substitution the current vector changes its sign properly in the q-number theory quantised according to the exclusion principle.

To make a transition to a neutral particle of spin 1/2 we now make an "abbreviation" of the theory (due to Majorana) by striking out w and its bracket relations or, which is the same thing, by identifying the charge conjugate states. Making this abbreviation *in the q-number theory*, it can be shown that the current vector vanishes identically as does also the magnetic moment. It is interesting to notice that the question is not yet settled as to whether the appropriate theory for the neutrino is the abbreviated theory or the unabbreviated one of a single real field.

The notion of charge conjugate states introduced above appears to have an algebraic significance in that it can be extended to

higher spins also. For such a generalisation the most appropriate field quantities to work with are the *undors*. For particles of spin 1/2 the field functions are undors of the first rank i.e. a Dirac spinor u_ρ, and the Majorana theory in which the abbreviation process is equivalent to identifying charge conjugate states deals therefore with self-charge-conjugated four spinors or what might be called *neutrettors of the first rank.* Similarly as has been shown by Belinfante a symmetric undor of the second rank $\psi_{k_1 k_2}$ (field functions in case of spin 1) can be associated with a charge-conjugated undor according to

$$\psi^{£}_{k_1 k_2} = £^{(1)} £^{(2)} (\psi_{k_1 k_2})^*$$

where £ is identical with the Pauli matrix C*, and $\psi^{£}$ is called the charge-adjoint of ψ. A *neutrettor of the second rank* can now be defined as a self-charge-adjoint undor of the second rank, and it can be shown that the field functions used in Kemmer's theory of neutral mesons are such neutrettors.

In the general case of arbitrary spin, the field is described by a set of undors $\psi_{k_1 k_2 \ldots k_n}$ which satisfy certain field equations in which some constants of the dimension of a charge may occur. By suitably defining a generalised matrix operator

$$£ = \prod_{j=1}^{n} £^{(j)}$$

which is a generalisation of the Pauli matrix C, we can operate by it on the conjugate complex $\psi^*_{k_1 k_2 \ldots k_n}$, and derive the transformed undor

$$\psi^{£} = £\psi^*$$

with the property that in the field equations satisfied by the $\psi^{£}$ the constants of the dimension of a charge, for instance e, are changed into their opposite:

$$e^{L} = -e^*$$

Further it can be shown that the transformation.

$$\psi \rightarrow \psi^{L} \text{ together with } e \rightarrow e^{L} \qquad \text{(A)}$$

where ψ^{L} is given by

$$\psi^{L} = \psi^{£} \qquad \text{(B)}$$

not only ensures the invariance of the field equations but also the invariance of all physically significant quantities (such as for example the total energy or the total charge) provided one works with a q-number theory i.e. *only on account of the commutation rules* holding

between the undor components. Thus, under the transformation specified by (A) and (B) we pass from one description of the field of particles and quanta to another completely equivalent one which, in the terminology of Kramers, can be called the charge-conjugated description of the physical world. The invariance of physical quantities under the transformation can also be called *charge invariance.*

Since this charge invariance exists on account of the commutation relations between undor components, it follows conversely that the postulation of charge invariance under (A) and (B) would provide information on the nature of the commutation rules, in particular, to decide between commutativity or anti-commutativity of the field components ψ and ψ^*. In fact, it can be shown that this postulation enables one to establish the connection between spin and statistics which has been set up earlier (§ 4) on other general considerations. That the statistical behaviour of elementary particles can be derived as a consequence of the postulate of charge invariance under (A)-(B) shows indirectly the partly algebraic nature of this concept.

A deeper analysis of the statistical behaviour of elementary particles under the assumption of charge invariance, and under other general assumptions, and the relationships between these assumptions has been made by Pauli and Belinfante. For this purpose they have generalised the notion of charge invariance by dropping the particular specification (B) of the transformation, and retaining only (A), calling it the *unspecified charge invariance.* The justification for this generalisation lies in the fact that (A) by itself, even without (B), would mean that there is a description of the physical world in which every "elementary charge" has the opposite sign, and the existence of such a generalised charge-conjugated description may perhaps be inferred from some speculations on a possible symmetry between positive and negative charges, which-though not existing on the earth-may still be a fundamental property of nature. The main point of the analysis is that, so far as the determination of commutativity or anti-commutativity of the field components is concerned, the postulate of unspecified charge invariance is of less restricted scope than the general assumptions used in § 4, viz. (1) there is no infinite number of negative energy states for the particles, (2) observables in points connected by a space-like vector are commutative. Even in the most general case where there are several possibilities of building up a quantised field theory (for eg. using more than one undor) the latter are two postulates of positive energy, and commutativity of observables in points connected by a space-like vector are sufficient

to decide between the + or − commutation bracket, while this is not true for the postulate of unspecified charge invariance.

It would also appear that the recent suggestion of Dirac to assume that probability coefficients calculated in a hypothetical world almost saturated with positrons are the same as those of the actual world is a form of the assumption of unspecified charge invariance.

§8. Subtraction formalism.

The theory set up in the previous articles appears satisfactory to deal with the general case as well as the special cases of zero rest mass and real field for the several spins. The basic facts are that

(i) in the case of half-integral spin the total charge can be made positive, and the hole theory removes the negative energy difficulty by quantisation according to Fermi-Dirac statistics,

(ii) in the case of integral spin the theory can be so formulated as to have the total energy positive, and the difficulty of negative particle density is removed by double quantisation according to Einstein-Bose statistics.

But there appear difficulties connected with infinities occurring in the mathematical formalism in both the cases. Thus in the first case the hole theory involves an infinite number of electrons and this necessitates the use of wave functions of great complexity, and the mathematics is so complicated that one cannot solve even the simplest problems accurately. Again in the second case, when one tries to solve the wave equations set up one gets divergent integrals in the solution and it would appear that the method fails. Thus in both the cases the infinities have got to be removed before a satisfactory theory can be built. Referring to Cantor and his school of workers on transfinite cardinal numbers, and the paradoxes arising therein, Poincaré said "The Cantoriens have created infinity, and are now being swallowed up by the demon of their own creation". The situation is analogous to what is found in modern quantum theory with this difference that the infinities are not the creation of quantum mechanics but are already there in classical theory itself. An attempt at the removal of infinities in classical mechanics has been made by Dirac using a method which, from the mathematical point of view, amounts to subtraction of an infinity from the infinity in the solution so as to get a finite result. This method, "the subtraction formalism" or "subtraction physics" was originally given only in classical mechanics, and in a recent paper

Dirac has shown that it is possible to embody it successfully in the quantum theory also. But the employment of this technique makes it necessary to revise some of the theories mentioned in the previous pages. For particles of half-integral spin a modification of the hole theory, as indicated towards the end of the previous article, becomes necessary, while for particles of integral spin the technique of making the energy positive definite has to be given up. The alternative formalism for the latter case as indicated by Dirac contains the point of view, novel from the mathematical stand point, that operators corresponding to real dynamical variables in the classical theory need be no longer self-adjoint. Pauli has recently worked out this formalism by using an indefinite metric in Hilbert space, and shows that this leads to some essentially new results for e.g. that operators with only positive eigenvalues can have negative expectation values. While the physical meaning of this is not clear it can be shown that it enables us to overcome the convergence difficulties of quantum electro-dynamics.

While algebra can be said to have successfully tackled the questions relating to spin, statistics, and perhaps charge, questions about mass, energy, and momentum require drastic improvements in the methods of mathematical analysis employed.

REFERENCES

(1) Bhabha — Presidential Address, Section of Physics, Indian Science Congress, 1943.

(2) Dirac — *Proc. Roy. Soc.* A., 180, 1942, p. 1-40.

(3) Fierz — *Helv. Phys. Acta.*, 12, 1939, p. 3-37.

(4) Kemmer — *Proc. Camb. Phil. Soc.* 39, 1943, p. 189-96.

(5) Pauli — *Rev. Mod. Phys.* 13, 1941, p. 203-232.

(6) „ — *Rev. Mod. Phys.* 15, 1943, 175.

(7) „ — *Phys. Rev.* 58, 1940, p. 716.

(8) „ & *Belinfante Physica.*, 7, 1940, p. 177.

(9) Schrodinger — *Proc. Roy. Irish. Acad.* A., 48, 1943, 135. See also

(10) *Proc. Ind. Acad. Sci.* 15, 1942, 139.

[*Note*.—Further references found in the text can be tak out from the references given above.]

Connection between Spin and Statistics

BY

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§ 1. *Definition of spin:*

In the relativistic field theory a particle is described by a wave-field, the wave-functions U (j, k) being tensors or spinors which transform according to irreducible representations of the Lorentz group. To obtain a definition of the spin of the particle we transform to the rest-system of the particle and in this system we reduce the field components into parts irreducible against space-rotations. To each such part with $r = 2f+1$ components belong r different eigen-functions which transform among themselves and belong to a particle with spin f. From the Lorentz invariance it follows that the same is true in an arbitrary reference system. If there exists in the rest-system only one irreducible group of components the field equations describe particles with only one spin-value, but this will not be the case in general unless certain additional conditions are imposed. The procedure becomes indeterminate in the case of particles with zero rest-mass but Pauli and Fierz have shown how it may be modified so as to obtain a consistent definition of spin in this case also. It is thus found that in an arbitrary reference-system and for arbitrary rest-masses, the U's which transform according to double valued representations with $j+k$ half-integral describe particles with half-integral spin, and those which transform according to single-valued representations with $j+k$ integral describe particles with integral spin.

In what follows it is not necessary to assume that the field describes particles with a single spin value, but only that they shall all be integral or half integral.

§ 2. *Indefiniteness of charge and energy respectively in the two cases* (A) *Integral Spin: $j+k$ integral.*

We divide the U's into two classes (i) the "+1 class" with j integral and k integral, (ii) the "−1 class" with j and k both half-integral. From the rules about the reduction of a product into irreducible constituents under the Lorentz group, it follows that the product of two U's of the same class always belongs to +1 class, while the product of two U's of two different classes belongs to the

-1 class. Further it is evident that the conjugate complex U^* of U belongs to the same class as U and that tensors of even rank belong to the $+1$ class, those of odd rank to the -1 class. In particular the propagation vector k_i belongs to the -1 class. We denote the U's belonging to the two classes by $U+$ and $U-$ respectively.

The field equations may be symbolically written

$$\sum kU^+ = \sum U^-;\ \sum kU^- = \sum U^+ \quad \text{... (1)}$$

with $k_i \equiv -i\frac{\partial}{\partial x_i}$, and omitting an even number of k-factors on either side since this does not affect the transformation-character.

These equations remain invariant under the substitution

$$\left.\begin{array}{l} k_i \rightarrow -k_i;\ U^+ \rightarrow U^+;\ U^- \rightarrow -U^- \\ (U^+)^* \rightarrow (U^+)^*;\ (U^-)^* \rightarrow -(U^-)^* \end{array}\right\} \quad \text{... (2)}$$

[It should be noticed that this is the only substitution of this kind which leaves the wave equations (1) invariant and is consistent with the rules for products referred to above.]

A tensor of even rank composed bilinearly or quadratically of the U's, being composed of quantities of the $+1$ class, must be of the form

$$T = \sum U^+U^+ + \sum U^-U^- + \sum U^+kU^- \quad \text{... (3)}$$

Here again an even number of k-factors is omitted and U, U^* are not distinguished since they both belong to the same class.

Under the substitution (2) $\quad T \rightarrow T$

On the other hand a similar tensor of odd rank, S, must be of the form

$$S = \sum U^+U^- + \sum U^+kU^+ + \sum U^-kU^- \quad \text{... (4)}$$

Under (2) $\quad S \rightarrow -S.$

In particular the current vector $s_i \rightarrow -s_i$.

The indefinite character of the current-density and the total charge for integral spin follows, for to every solution there belongs another for which the components of s_i change sign. The definition of a definite particle-density which transforms like the 4th component of a 4-vector is thus impossible.

(B) *Half-integral spin: $j+k$ half-integral:*

Here we classify the U's as follows.

"$+\varepsilon$ class" : j integral, k half-integral

"$-\varepsilon$ class" : j half-integral, k integral.

The multiplication table for the 4 classes ± 1, $\pm \varepsilon$ is as follows:

	+1	-1	+ε	-ε
+1	+1	-1	+ε	-ε
-1	-1	+1	-ε	+ε
+ε	+ε	-ε	+1	-1
-ε	-ε	+ε	-1	+1

It will be observed that this multiplication table is the same as that of Klein's 'four-group' ("*vierergruppe*"). It will be recalled that the latter is a normal subgroup of the alternating group A_4 of 4 letters, and has the elements (1), (12)(34), (13)(24), (14)(23). If we write $1, -1, \varepsilon, -\varepsilon$ for these with $\varepsilon^2 = 1$, we get the above table.

Here $(U^{+\varepsilon})^*$ belongs to $-\varepsilon$ class and $(U^{-\varepsilon})^*$ belongs to $+\varepsilon$ class and it is therefore necessary to distinguish U and U*

Instead of (1) we have here the field equations

$$\left.\begin{aligned}\sum kU^{+\varepsilon} + \sum k(U^{-\varepsilon})^* &= \sum U^{-\varepsilon} + \sum (U^{+\varepsilon})^* \\ \sum kU^{-\varepsilon} + \sum k(U^{+\varepsilon})^* &= \sum U^{+\varepsilon} + \sum (U^{-\varepsilon})^*\end{aligned}\right\} \quad \ldots (5)$$

with the same convention as to k-factors as before.

These equations remain unaltered under the substitution

$$\left.\begin{aligned}&k_i \rightarrow -k_i,\ U^{+\varepsilon} \rightarrow iU^{+\varepsilon};\ U^{-\varepsilon} \rightarrow -i\, U^{-\varepsilon} \\ &(U^{-\varepsilon})^* \rightarrow i(U^{-\varepsilon})^*;\ (U^{+\varepsilon})^* \rightarrow -i\,(U^{+\varepsilon})^*\end{aligned}\right\} \quad \ldots (6)$$

Under this substitution it is easily verified that a tensor of even rank T composed bilearly or quadratically of the U's goes over into its negative, while a similar one of odd rank remains unaltered. Thus in the case of half-integral spin a positive definite energy-density and total energy is impossible.

§ 3. *Quantisation—Connection between spin and statistics.*

Our considerations so far have been restricted to the c-number theory. But the impossibility of defining a positive definite charge in the case of integral spin, and of a positive definite energy in the case of half-integral spin shows the need for transition to the

q-number theory. We postulate that all physical quantities at finite distances exterior to the light cone ($|x_0' - x_0''| < |X' - X''|$) are commutable. In the simplest case of the absence of interaction, the bracket expressions of all quantities satisfying the wave equation

$$\Box U - \chi^2 U = 0 \qquad \ldots (7)$$

may be expressed by means of a function D and a finite number of its derivatives, where

$$D = \frac{1}{(2\pi)^3} \iiint \exp i\,(kX)\, \frac{\sin k_0 x_0}{k_0}\, d^3k$$

$$k_0 = +\sqrt{\chi^2 + k^2} \qquad \ldots (8)$$

It may be proved that D vanishes outside the light-cone $-x_0 < r < x_0$. Assuming the U's to be ordered in such a way that each field component is composed of quantities of the same class, we have

$[U^{(r)}(X', x_0'), U^{*(r)}(X'', x_0'')]$ belongs to -1 class for half-integral spin and to $+1$ class for integral spin, and hence the bracket expression must consist of derivatives of odd order of D $(X' - X'', x_0' - x_0'')$ for half-integral spin, and of derivatives of even order of D for integral spin. Consider the expression

$$F = [D^{(r)}(X', x_0'), U^{*(r)}(X'', x_0'')] + [U^{(r)}(X'', x_0''), U^*(X', x_0')] \qquad \ldots (9)$$

Observing that D is even in space-coordinates, odd in time coordinate, and that F is symmetric, we see that F must consist of derivatives of D which are of even order *w. r. t.* space and of odd order *w. r. t.* time. This is consistent with the above for half-integral spin, but not for integral spin unless $F = 0$ (for integral spin).

Now for quantisation with the Einstein-Bose statistics we use the bracket $[A, B]^- = AB - BA$, while with the Fermi-Dirac statistics we use $[A, B]^+ = AB + BA$. By inserting $[AB]^+$ in $F = 0$ we obtain a contradiction, since the l. h. s. is essentially positive for $X' = X''$, $x_0' = x_0''$. Hence we have the result that for integral spin quantisation according to the exclusion principle is impossible. On the other hand, for half integral spin it is formally possible to quantise according to the Einstein-Bose statistics, but the energy of the system would not be positive. The idea of negative energy is not physically admissible and can be avoided only by some such theory as Dirac's theory of holes which requires the exclusion principle.

The above general proof of the connection between spin and statistics is due to Pauli,† according to whom it constitutes one of the most important applications of the special relativity theory.

† Pauli: *Physical Review*, Vol. 58 (1940).

SYMPOSIUM ON TOPOLOGY

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TOPOLOGY AND FUNCTIONAL EQUATIONS

The Method of Leray—Schauder

BY

REV. FATHER C. RACINE, *Loyola College, Madras.*

Brouwer has defined the *topological degree* of a continuous transformation in an n-space. In the case of a subset C of a function space E which is compact and is the transform of an open and bounded set T by means of a continuous transformation, the notion of topological degree can be easily defined also.

In particular if E is a Banach space and F (x) a completely continuous (vollstetig) transformation, the topological degree of the transformation $g = x - \mathrm{F}(x)$ can be determined at the point $y = 0$.

Let us now consider the functional equation.

(1) $x - \mathrm{F}(x, k) = 0.$

where

(*a*) x and F belong to a Banach space E,

(*b*) $k_1 \leqslant k \leqslant k_2$, where k_1, k, and k_2 are real,

(*c*) F (x, k) considered as function of the point (x, k) in the space $[\mathrm{E} \times \mathrm{K}]$ (and defined also on its closure) is completely continuous over that closure, and uniformly continuous in k,

(*d*) The boundary Ω' of $\Omega = [\mathrm{E} \times \mathrm{K}]$ does not contain any solution (x, k) of (1)

Then it can be proved that if for a value $k=k_0 \in (k_1, k_2)$ the topological degree of the transformation $g=x-\text{F}(x, k)$, for $y=0$, is different from zero, it is also different from zero for all $k \in (k_1, k_2)$, and it is an invariant of the transformation. Now if the equation $x-\text{F}(x, k_0)=0$ has got one or an odd number of solutions, its topological degree (or index) for $k \in (k_1\ k_2)$ will be different from zero. It is then readily seen than there is at least one solution to the equation (1) for $k \in (k_1\ k_2)$. Thus we have a theorem of existence.

Fractional Powers of Linear Operators

BY

M. V. SUBBA RAO, *University of Madras.*

A systematic account of the existence of fractional powers of linear operators is given under the following conditions: let U be a linear operator mapping a Banach Space on itself and let I be the identical transformation. We take the linear operator $\text{T}=\text{I}+\lambda\text{U}$, ($\lambda$ a constant). Every positive integral power of T exists. We prove the following results:—

(A) T generates a group when the inverse of T exists.

(B) T generates a continuous group, if $\|\text{U}\| < \frac{1}{|\lambda|}$.

We can sharpen the latter result in the case when U is a continuous Hermetian Symmetric positive definite operator defined in a Hilbert Space. We then prove the following result:—

(C) If $|\lambda|$ is less than every eigenvalue of U, then T generates a continuous group.

A corresponding result can be stated for non-symmetric operators U:—

(D) If $|\lambda|$ is less than every eigen value (assumed positive) of the Symmetric operator UU* (U* being the adjoint of U), then T generates a continuous group.

The above general results can be applied to special cases, like the Fredholm operator

$$\text{T}\, f(x)=f(x)+\lambda \int_a^b \text{K}\,(x, y)\, f(y)\, dy; \quad f(x) \text{ and } \text{K}\,(x, y) \in \text{L}^2.$$

When the upper limit of integration is a variable, we prove the following results.

(E) The operator $\mathrm{T}\, f(x)=f(x)+\lambda \int_a^x \mathrm{K}\,(x, y)\, f(y)\, dy$ generates a continuous group for all λ (a being finite).

(F) If the kernel $\mathrm{K}\,(x, y)$ has the form $\mathrm{K}\,(x, y)=\alpha\,(x)\cdot\beta\,(y)$, then for the operator $\mathrm{T}\, f(x)=\int_a^x \mathrm{K}\,(x, y)\, f(y)\, dy$; $f(x)$, $\mathrm{K}\,(x, y)\cdot \in \mathrm{L}^2$, every positive power is defined and is given by

$$\mathrm{T}^n\, f(x)=\frac{\alpha\,(x)}{\Gamma\,(n)}\cdot \int_a^x \left[\int_y^x \alpha(t)\ \beta(t)\ dt\right]^{n-1} \beta(y)\cdot f(y)\ dy.$$

The case of the fractional integral arises when $\mathrm{K}\,(x, y)=1$.

We finally state the following result due to Volterra, for the operator T defined above in (F).

(G) If $k\ (x > y)$ is of finite order α, and if it is possible to determine a function of order $\frac{\alpha}{n}$, (n being a given integer), and permutable with $\mathrm{K}\,(x, y)$ then $\mathrm{T}^{1/n}$ exists for that value of n.

Linear equations in a quaternionic field

BY

AFZAL AHMED, *Osmania University.*

E. Study was probably the first mathematician who studied the system of linear equations

$$\sum_{k=1}^{n} a_{ik}\ x_k = y_i \qquad (i=1,\ 2,\ .\ .\ .\ n.)$$

where all the elements a_{ik}, x_k and y_i are quaternions. H. Wachs in 1934 conceived a space, analogous to Hilbert space, whose vectors have quaternions as their components.

The author has attempted to solve the system of linear equations above with the help of quaternionic matrices by defining the determinant of a self-adjoint quaternionic matrix, and also employing the notion of vector space. The system of infinite linear equations in an infinite number of unknowns have also been solved under suitable conditions.

The Topology of Ordered spaces

BY

A. SURYANARAYANAMURTY, *University of Madras.*

If S is a topological space and S_r the space of ordered r-ads of distinct elements of S, then the smallest value of r for which S_{r+1} is disconnected, is defined, here to be the connection-index of S. S is said to have a cut r-ad, if the removal of r distinct elements from S, disconnects S. If S has a cut r-ad but has no cut $(r-1)$-ad, then S is said to have cut-index r and is denoted, here by r_2. The remarkable properties of these indicies are as follows:

(i) If $r_2=1$, $r_1=1$ or ∞; (ii) If $r_2=2$, $r_1=2$ or ∞

(iii) If $r_2>2$, $r_1=\infty$; (iv) For linear spaces $r_1=r_2=1$.

(v) For cyclic spaces $r_1=r_2=2$.

These results show that linear and cyclic spaces are the only ones that have finite values, for their connection—index. Thus, their unique topological status, among spaces, in general, is established.

REVIEWS

1. *Text-book on Differential Calculus* (Price Rs. 5/-)
2. *Text-book on Integral Calculus* (Price Rs. 5/-)
3. *Key to Differential Calculus* (Price Rs. 2/8)
4. *Calculus for Intermediate Students* (Price Re. 1/-)

By Dr. Gorakh Prasad, Reader in Mathematics, Allahabad University. [1, 2, 3 are published by The Benares Mathematical Society, while the last book is published by Ram Narain Lal, Allahabad].

The first two books together meet a long felt need for a text-book on Calculus covering completely the syllabus for the B. A. and B. Sc., pass degree courses of the various Indian Universities. The treatment of the subject is lucid and thorough covering every detail of the various topics discussed. There are a good number of worked examples. The two books contain a rich variety of problems for the would be pupil to exercise his ingenuity upon. The third book contains the solutions of the problems in the first book.

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V. G.